Published by Institute of Physics Publishing for SISSA

Received: *April 28, 2008*
Accepted: *June 12, 2008*
Published: *June 23, 2008*

Semiclassical quantisation of finite-gap strings

Benoît Vicedo

DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.
E-mail: bv213@damtp.cam.ac.uk

Abstract: We perform a first principle semiclassical quantisation of the general finite-gap solution to the equations of a string moving on $\mathbb{R} \times S^3$. The derivation is only formal as we do not regularise divergent sums over stability angles. Moreover, with regards to the AdS/CFT correspondence the result is incomplete as the fluctuations orthogonal to this subspace in $AdS_5 \times S^5$ are not taken into account. Nevertheless, the calculation serves the purpose of understanding how the moduli of the algebraic curve gets quantised semiclassically, purely from the point of view of finite-gap integration and with no input from the gauge theory side. Our result is expressed in a very compact and simple formula which encodes the infinite sum over stability angles in a succinct way and reproduces exactly what one expects from knowledge of the dual gauge theory. Namely, at tree level the filling fractions of the algebraic curve get quantised in large integer multiples of $\hbar = 1/\sqrt{\lambda}$. At 1-loop order the filling fractions receive Maslov index corrections of $\frac{1}{2}\hbar$ and all the singular points of the spectral curve become filled with small half-integer multiples of $\hbar$. For the subsector in question this is in agreement with the previously obtained results for the semiclassical energy spectrum of the string using the method proposed in hep-th/0703191. Along the way we derive the complete hierarchy of commuting flows for the string in the $\mathbb{R} \times S^3$ subsector which are generated by the Taylor coefficients of the quasi-momentum $p(x)$ through Hamilton's equation. Moreover, we also derive a very general and simple formula for the stability angles around a generic finite-gap solution which may be used in the study of stability properties of solutions in the $\mathbb{R} \times S^3$ subsector. We also stress the issue of quantum operator orderings and whether or not a given ordering preserves integrability since this problem already crops up at 1-loop in the form of the subprincipal symbol.

Keywords: Bosonic Strings, Integrable Hierarchies, Integrable Field Theories, AdS-CFT Correspondence.

JHEP06(2008)086

Contents

1. Introduction

The method of semiclassical quantisation in field theory has been extensively developed by many authors in the 70's using different approaches [1 – 7] (see also the books [8, 9] for a more or less complete survey and list of references). The aim of all these methods is to give a quantum mechanical meaning to extended classical solutions of the field equations which already classically exhibit particle like properties. The role played by such non-trivial classical solutions in the leading order quantisation of any field theory is evident from the path integral which is dominated by classical solutions in the $\hbar \to 0$ limit. It follows then that the applicability of semiclassical methods crucially relies on an explicit knowledge of

JHEP06(2008)086

classical solutions. Yet for a generic field theory, very little can be said about explicit solutions to the field equations and in most cases a general solution does not exist. When the field theory is classically integrable however, essentially everything is known about the classical theory and the most general solution can be constructed explicitly in terms of standard functions and finitely many algebraic operations. In this case the complete semiclassical spectrum of the theory can then be obtained by applying the methods of semiclassical quantisation to the general solution.

It is now very well established that the Metsaev-Tseytlin action [10] describing superstrings on $AdS_5 \times S^5$ is classically integrable [11], in the sense that the theory possesses an infinite number of integrals of motion. This fact has been thoroughly exploited in the literature [12] to completely classify the full set of classical solutions on $AdS_5 \times S^5$ by assigning to every solution a finite genus algebraic curve which encodes its integrals of motion $I_1, \ldots, I_n$. However, the algebraic curve is not enough to uniquely specify the solution, which can be seen as follows. Since a given solution carries only finitely many non-zero integrals of motion $I_1, \ldots, I_n$ it will be invariant under the action of all the other integrals of the theory. Moreover, the solution breaks all the symmetries generated by $I_1, \ldots, I_n$ and the action of these integrals on the solution will generate new solutions with the same integrals. Indeed, in the theory of finite-gap integration [30 – 36], (finite-gap) solutions are shown to be in one-to-one correspondence with sets of algebro-geometric data which essentially consist of a finite genus algebraic curve equipped with a finite set of points called a *divisor*. The action of the moduli $I_1, \ldots, I_n$ of a solution on the solution itself will act non-trivially on the divisor, thereby generating a new solution with different divisor. The divisor therefore encodes the different zero-modes of a given solution.

The treatment of the zero-modes is an important part of any approach to semiclassical quantisation [1, 2, 8, 9]. Indeed, if a classical solution has zero-modes then a naive semiclassical quantisation of the solution will fail. Consider a solution $\phi_{\rm cl}$ of a field equation derived from an action $S[\phi]$, i.e. $S'[\phi_{\rm cl}] = 0$, where $'$ denotes $\delta/\delta\phi$. If v denotes an infinitesimal symmetry of the equations of motion, i.e. $v(S'[\phi]) = S''[\phi](v\phi)$, and suppose that $\phi_{\rm cl}$ is not invariant under the symmetry then it follows immediately that $(v\phi_{\rm cl}) \neq 0$ is in the kernel of the operator $S''[\phi_{\rm cl}]$ which is therefore not invertible and so the propagator of the theory in the background $\phi_{\rm cl}$ cannot be defined. The standard way around this difficulty is to treat the zero-mode directions separately using the method of 'collective coordinates'. In short, collective coordinates parametrise the zero-mode directions, namely the flat directions in field space, along which the wave function will tend to spread out in the form of a plane wave as a result of which the quantum counterpart of the solution $\phi_{\rm cl}$ will acquire dynamics along the collective coordinates. Generally one has to perform a change of variables in field space to include the collective coordinates among the set of field variables and this can often only be done implicitly. A nice feature of the finite-gap construction is that it naturally lends itself to the separation of zero-modes since the divisor, which plays the role of the collective coordinates, already appears explicitly in the finite-gap solution – no change of variables was required.

The divisor of a finite-gap solution therefore plays a central role in determining its semiclassical spectrum. But although the algebraic curve is known in full generality for the

JHEP06(2008)086

$AdS_5 \times S^5$ superstring, the divisor has only been identified so far in the subsector $\mathbb{R} \times S^3$ for which the explicit reconstruction of finite-gap solutions from the algebro-geometric data has been studied [13 – 15]. The method of semiclassical quantisation as stated above can therefore only be applied directly in the subsector $\mathbb{R} \times S^3$. In this paper we perform such a semiclassical analysis of bosonic string theory on $\mathbb{R} \times S^3$ from first principles. We do not attempt to include the fluctuations in the directions transverse to the subspace $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$ for clarity and because we believe that the method presented here should carry over with few alterations to the full case of superstrings on $AdS_5 \times S^5$ once the divisor is known. The calculation therefore serves as a toy model for understanding from the finite-gap perspective the origin of the discretisation of the algebraic curve when leading order semiclassical corrections are included. Nevertheless, our result agrees for fluctuations within the $\mathbb{R} \times S^3$ subsector with the semiclassical results of [16 – 18].

In the remainder of the introduction we start by recalling the method of semiclassical quantisation à la Dashen, Hasslacher and Neveu [1 – 3] when applied to the specific example of the breather solution in Sine-Gordon theory. We reformulate everything in a language that we hope will facilitate the conceptual understanding of the method in the finite-gap setting and in the last part of the introduction we give a sketch of the ideas developed in the paper.

1.1 Semiclassical Sine-Gordon breathers

Consider the example of the boosted breather solution in Sine-Gordon theory [2, 8, 9]

$$\phi_{\tau,v}(x,t) = \frac{4m}{\sqrt{\lambda}} \tan^{-1} \left\{ \frac{((\tau m/2\pi)^2 - 1)^{\frac{1}{2}} \sin[(2\pi/\tau)(t - vx)/(1 - v^2)^{\frac{1}{2}}]}{\cosh[((\tau m/2\pi)^2 - 1)^{\frac{1}{2}} (2\pi/\tau)(x - vt)/(1 - v^2)^{\frac{1}{2}}]} \right\}. \tag{1.1}$$

This is really a two parameter family of solutions parametrised by their proper period τ and their velocity v, or equivalently by their energy E and momentum p. To compute the (possibly continuous) spectrum of the corresponding quantum states it is always simpler at first to put the system in a very large but finite box of length L by identifying $x \sim x + L$ so as to make the spectrum discrete, and then take the infinite volume limit $L \to \infty$ at the end. In this closed-loop world the breather solution (1.1) is periodic in t of period T provided τ and v satisfy $T = l\tau/(1 - v^2)^{\frac{1}{2}} = mL/v$ with $l, m \in \mathbb{N}$.

If we were quantising the kink, we could move to its rest frame in which it is static and study small fluctuations in terms of eigenfrequencies. However, the breather is a little more complicated since it is time dependent in its rest frame, and because time dependent solutions are not point-like in field space, we need a way to characterise perturbations of the orbit as a whole. As we will describe in appendix B, this is done by considering the perturbation of a specific point on the orbit, evolving that perturbation under the equations of motion for roughly the period of the underlying solution, and comparing the final perturbation with the original one. If the perturbation is stable then it will have merely rotated and the angle of rotation is called the *stability angle*. If instead the perturbation is unstable it will have grown exponentially in magnitude, which corresponds to the case of a complex stability angle. Finally, if the perturbation comes back exactly to itself, this means

JHEP06(2008)086

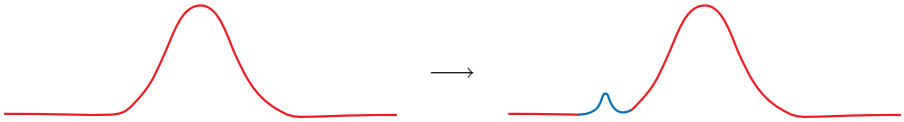


Figure 1: Perturbing the breather by another small breather using the Bäcklund transform

it describes a nearby periodic solution, and in general zero stability angles correspond to symmetries. In the case of the Sine-Gordon breather we therefore need to look for generic nearby solutions $\phi(x,t) = \phi_{\tau,v}(x,t) + \delta\phi$. This perturbed solution won't be periodic in general, yet because the linearised equation

$$\Box\delta\phi = (\cos\phi_{\tau,v})\,\delta\phi \tag{1.2}$$

is invariant under time translation by T we can always write its solution as a superposition of eigenfunctions of time translation $\delta\phi(x,t+T) = e^{-i\nu}\delta\phi(x,t)$, where ν are their stability angles. Notice that the Sine-Gordon equation is invariant under arbitrary space and time translations, but the breather solution $\phi_{\tau,v}$ is not. As a result, $\partial\phi_{\tau,v}/\partial x$ and $\partial\phi_{\tau,v}/\partial t$ are both *zero-modes*, i.e. perturbations with zero stability angles. In general, any symmetry of the action that is not a symmetry of the classical solution will give rise to a zero-mode.

The task of finding nearby solutions to the breather is greatly facilitated by the fact that the Sine-Gordon equation is integrable, since we can use the Bäcklund transform to get new solutions from known solutions. In particular we can perturb our breather by adding a little breather of small amplitude on top of it (figure 1). Studying double breather solutions in the limit where the small breather has vanishingly small amplitude corresponds to a linearised study of the Sine-Gordon equation around the breather solution. So integrability gives us a convenient way of writing down explicit solutions to the linearised equation (1.2) from which the stability angles of the breather may be read off.

Identifying the space of classical solutions with phase-space, for each τ, v (or equivalently E, p) the breather solution (1.1) is just a specific point in phase space. However, the existence of two zero-modes $\partial\phi_{\tau,v}/\partial x$ and $\partial\phi_{\tau,v}/\partial t$ for the breather solution indicates that it really belongs to a two parameter family of solutions with the same integrals of motion E, p. These are the space and time translated breather solutions

$$\phi_{\tau,v}(x+x_0, t+t_0). \tag{1.3}$$

Since all the other stability angles of the breather are real, when we include first order quantum corrections the wavefunction will want to localise around not one breather, but around the whole two parameter family (1.3) of breathers by spreading along the flat directions, namely the x_0 and t_0 directions. Along these directions the wavefunction will therefore be a plane wave, but since the t_0-direction is closed by periodicity of the breather solution the plane wave along it must have an integer number of peaks and troughs. In other words the change of phase of the wavefunction around this closed direction will have to be an integer multiple n of 2π. Along all the other non-zero stability angle directions the wavefunction will decay rapidly and, intuitively, for states with higher excitation number n_i it will extend further in these directions. The correct quantisation conditions encoding the

JHEP06(2008)086

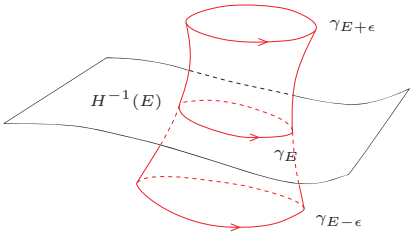


Figure 2: Cylinder theorem: a periodic solution γ_E on the energy level $H^{-1}(E)$ is contained in a one parameter family of periodic solutions of varying energy in the range $[E-\epsilon, E+\epsilon]$.

semiclassical energy spectrum of the wavefunction localised around the family of breather solutions was first derived by Dashen, Hasslacher and Neveu [1] and can be expressed as follows. If we define the 'action' of the breather solution as

$$W(E) = \int_0^T dt \int dx \pi_{\tau,v}(x,t)\partial_0\phi_{\tau,v}(x,t), \tag{1.4a}$$

then the DHN quantisation conditions read

$$\frac{W(E)}{\hbar} = 2\pi n + \sum_{\nu_i>0}\left(n_i + \frac{1}{2}\right)\nu_i + O(\hbar). \tag{1.4b}$$

Although the derivation of this formula is very complicated, it intuitively makes a lot of sense. In general the phase of the wavefunction in the semiclassial approximation is an action of the form (1.4a) so the first term on the right hand side of (1.4b) can be seen to comes from the single-valuedness of the wavefunction along the compact t_0-direction whereas the correction from the sum over stability angles is related to the small fluctuations transverse to the t_0 and x_0 directions.

For the purpose of drawing the analogy between the Sine-Gordon breather case here and that of finite-gap strings discussed later it will be convenient to think of the conditions (1.4) in more geometric terms in phase-space as follows. Since the breather in (1.3) with $x_0 = 0$ is periodic, it can be thought of as a closed orbit on the level set $\Sigma_{E,p}$ of fixed E, p. The direction along the orbit, parametrised by t_0, corresponds to the zero-mode $\partial\phi_{\tau,v}/\partial t$ of the breather. But since it has another zero-mode, namely $\partial\phi_{\tau,v}/\partial x$, this orbit really belongs to a continuous family of periodic orbits, parametrised by x_0, all contained in $\Sigma_{E,p}$. However, because we are working in a periodically identified finite box, this two parameter (x_0, t_0) family of breathers is in fact a torus $\mathbb{T}^2_{E,p}$ lying within $\Sigma_{E,p}$. And since all the other stability angles of the breather are non-zero, this means that $\mathbb{T}^2_{E,p}$ is isolated on the level set $\Sigma_{E,p}$ in the sense that it does not belong to a larger continuous family of periodic orbits within $\Sigma_{E,p}$. Yet if we leave the level set $\Sigma_{E,p}$, one can show that in a neighbourhood of $\Sigma_{E,p}$ the torus $\mathbb{T}^2_{E,p}$ persists, namely it belongs to a two parameter family of torii parametrised by E, p. This is the content of the 'cylinder theorem', illustrated in figure 2 for the case of a solution with a single zero-mode, so that its zero-mode family in the level set $H^{-1}(E)$ is just a circle S^1_E that belongs to a cylinder $S^1_E \times [E-\epsilon, E+\epsilon]$. Looking back at the most general breather solution (1.3) it contains four independent parameters:

JHEP06(2008)086

the two parameters x_0, t_0 are parameters along the torus $\mathbb{T}^2_{E,p}$ whereas E, p parameterise the family of torii of the cylinder theorem. Now the effect of the quantisation condition (1.4) is to pick out a discrete set of breathers from this 'cylinder' of breathers (1.1), the energy and momentum of which approximate to order $O(\hbar)$ the semiclassical energy spectrum of the quantum states localised around the breather solution. For instance, when applied to the Sine-Gordon breather the quantisation conditions (1.4) yield the following semiclassical spectrum [2]

$$E_{k,n} = (p_k^2 + M_n^2)^{\frac{1}{2}}, \quad p_k = \frac{2\pi k}{L},$$

where $M_n = \frac{16m}{\gamma'} \sin \frac{n\gamma'}{16}$ and $\gamma' = \frac{\lambda}{m^2}\left(1 - \frac{\lambda}{8\pi m^2}\right)^{-1}$, and in the infinite volume limit $L \to \infty$ the momentum becomes continuous as expected.

1.2 Sketch of semiclassical finite-gap strings

We would like to apply a similar kind of reasoning to the case of superstring theory on $AdS_5 \times S^5$. However, since this formalism requires the knowledge of explicit solutions we will restrict attention to bosonic string theory on $\mathbb{R} \times S^3$ for which the general finite-gap solution to the equations of motion is known [13, 14]. In conformal static gauge the string is given by an embedding $g(\sigma, \tau) \in \mathrm{SU}(2)$ of the worldsheet into SU(2), and if we define the corresponding Lie algebra current $j = -g^{-1}dg \in \mathfrak{su}(2)$ then the equations of motion and Virasoro constraints take the following form

$$d * j = 0, \quad dj - j \wedge j = 0, \quad \frac{1}{2}\,\mathrm{tr} j_\pm^2 = -\kappa^2. \tag{1.5}$$

As is well know, the equations of motion are integrable and can be rewritten in the form of a zero-curvature equation $dJ(x) - J(x) \wedge J(x) = 0$. In this form one can make use of the powerful methods of finite-gap integration to construct, at least abstractly the general finite-gap solution to the equations of motion. In fact, it is possible to incorporate the Virasoro and static gauge constraints into the constructions [13, 14] so as to get only physical motions of the string. The general finite-gap solution is constructed from the following piece of *algebro-geometric data*:

- An algebraic curve [12] of genus g.
- A set of $g + 1$ points [13] on this curve.

Essentially, by the Riemann-Roch theorem there is an injective map from this algebro-geometric data into the space of solutions to (1.5). The idea of finite-gap integration is illustrated in figure 3: Every finite-gap solution to (1.5) is in one-to-one correspondence with an algebraic curve (of genus three in figure 3) equipped with a set of marked points (four of them in figure 3). The algebraic curve encodes the integrals of motion of the solution, and these points encode the dynamics. Their exact motion on the algebraic curve is very complex, but what we find is that if we map the algebraic curve to its (generalised) Jacobian, a $(g + 1)$-torus, via the (generalised) Abel map then the motion in σ and τ

JHEP06(2008)086

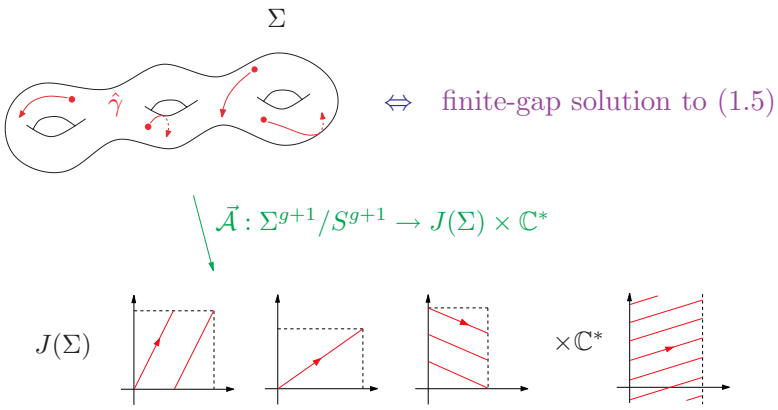


Figure 3: Idea of finite-gap integration.

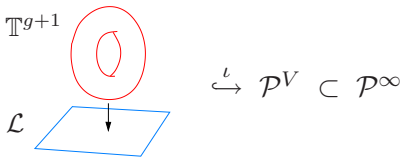


Figure 4: The algebro-geometric data as a $(2g+2)$-dimensional phase-space.

becomes extremely simple, namely it linearises. The motion of the string on this $(g+1)$-torus is like that of an infinitely rigid string wrapping one cycle of the torus and moving linearly in time along another direction.

An alternative way of picturing what a finite-gap solution looks like that will be useful later is as follows. As we just saw, the dynamics of a finite-gap solution corresponds to linear motion on a $(g+1)$-torus, which is very reminiscent of a finite-dimensional integrable system. In fact one can view the Jacobian as the Liouville torus of a $(2g+2)$-dimensional dynamical system. The base space $\mathcal{L}$ of this $(2g+2)$-dimensional system is the moduli space of the algebraic curve parametrised by the *filling fractions* $\{\mathcal{S}_I = \int_{\mathcal{A}_I} z dp\}_{I=1}^{g+1}$. But if the algebro-geometric data is to be thought of as a finite-dimensional phase-space it must be equipped with a natural symplectic structure. This can be obtained as follows: the finite-gap solution maps this algebro-geometric data to the space of solutions to (1.5), see figure 4. Identifying the space of solutions to the equations of motion with the phase-space $\mathcal{P}^\infty$, the solutions to (1.5) which also satisfy Virasoro and static gauge define a second class constraint surface $\mathcal{P}^V \subset \mathcal{P}^\infty$. This is equipped with a Dirac bracket induced by the Poisson bracket on the $\mathfrak{su}(2)$ current appropriately regularised à la Maillet [19, 14]. If one then pulls back this Dirac bracket to the algebro-geometric data using the finite-gap solution we obtain a 'natural' symplectic structure on the algebro-geometric data which can be concisely written as (see [14] for details)

$$\omega = \sum_{I=1}^{g+1} d\mathcal{S}_I \wedge d\varphi_I .$$

JHEP06(2008)086

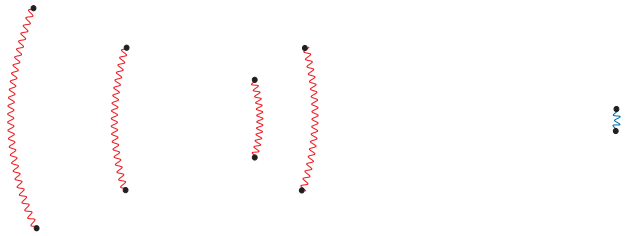


Figure 5: Perturbation of a finite-gap solution.

The upshot of this is that the filling fractions are precisely the action variables of the finite-gap string. They are the analogues of the period τ and velocity v (or energy E and momentum p) of the generic breather (1.3) which defined a four parameter family of solutions. A finite-gap solution defines a whole $(2g+2)$ parameter family of solutions parametrised by the algebro-geometric data and can be written as follows

$$g = g\Big(\sum\nolimits_N t_N \boldsymbol{U}_N(\boldsymbol{S}) + \boldsymbol{D}\Big|\boldsymbol{S}\Big),$$

where t_N are a set of $g+1$ independent times (defined in section 3), $\boldsymbol{U}_N(\boldsymbol{S})$ is a certain function of action variables $\boldsymbol{S}$ which play the role of the parameters (τ, v) here and $\boldsymbol{D} \in \mathbb{C}^{g+1}$ is the exact analogue of the initial coordinates of the breather (x_0, t_0). We therefore expect a finite-gap solution constructed from a curve of genus g to have $g+1$ zero-modes corresponding to the $g+1$ components of the vector $\boldsymbol{D}$.

In view of applying a semiclassical quantisation formula like the one in (1.4) we must first determine all the stability angles of a given finite-gap solution. So just as in the case of the Sine-Gordon breather, we would like to study perturbations of finite-gap solutions described above. Once again integrability will play a prominent role in solving the linearised equations. In fact, finding solutions to the linearised problem is very simple now that we have already fully exploited integrability to construct the most general finite-gap solution. A perturbation of a given finite-gap solution will simply be another 'nearby' finite-gap solution. Recall [13, 14] that in the SU(2) sector the algebraic curve is hyperelliptic and can be represented by a set of $g+1$ cuts in the complex plane. How can one describe perturbations of the g-gap solution corresponding to this curve? Playing the same game as for the Sine-Gordon breather where we used integrability to add another little breather on it, here we can just take a solution corresponding to a curve of genus one higher, but make the extra filling fraction very small, which corresponds to making the cut very small, see figure 5. There is an obvious analogy here between breathers in Sine-Gordon and cuts in bosonic strings on $\mathbb{R} \times S^3$ as one can think of a finite-gap solution as a multi-breather solution consisting of finitely many breathers. Cuts with small filling fractions are analogous to breathers of small amplitude as both describe perturbations. If we define the a_i-cycle $(i = 1, \ldots, g)$ as usual to encircle the i^{th} cut counterclockwise (on the upper sheet) then a perturbation of this kind clearly corresponds to pinching an a-cycle of the algebraic curve. So we want to take the difference between the solution before pinching an a-cycle and the solution after pinching the a-cycle; this will give us a perturbation of the latter and we can then analyse its periodicity properties to extract the corresponding stability angles.

JHEP06(2008)086

Notice however that any given perturbation of a finite-gap string will have one stability angle defined for each cycle on the Jacobian, or equivalently for each macroscopic cut.

The semi-classical spectrum can be obtained by performing a WKB analysis of the wavefunction that will localise around the zero-mode directions of the solution, which in the case of the finite-gap string is the Jacobian. Again, the leading term will describe how many full waves fit on the compact Jacobian, and the infinite sum corresponds to small fluctuations transversal to the Jacobian. The result of such an analysis that will be sketched in section 2.2 are the following set of Bohr-Sommerfeld equations, the correct form of which involves Maslov indices[1]

$$\frac{S_I}{\hbar} = N_I + \frac{\mu_I}{4} + \sum_{\alpha=g+2}^{\infty} \left(n_\alpha^I + \frac{1}{2} \right) \frac{\nu_\alpha^{(I)}}{2\pi} + O(\hbar). \tag{1.6}$$

Here $\mu_I = 2$ is the *Maslov index* of the $\mathcal{A}_I$-cycle ($I = 1, \ldots, g+1$) in the generalised Jacobian $J(\Sigma, \infty^\pm)$. Note that (1.6) is only valid in the harmonic oscillator approximation $N_I \gg n_\alpha$ where the perturbations are much smaller than the background filling fractions. So the expression (1.6) really contains two different orders, namely the tree level and 1-loop level of order $O(1)$ and $O(\hbar)$ respectively (after multiplying (1.6) throughout by $\hbar$). At tree level (1.6) simply expresses the fact that the filling fractions are quantised in integer multiples of $\hbar$, i.e. $S_I = N_I \hbar$, which is a straightforward consequence of the fact that the S_I are the action variables as was shown in [14]. The non-trivial content of (1.6), first obtained in the series of papers [16 – 18] and that we derive from first principles in this paper, is the 1-loop correction which includes firstly the Maslov index correction $\frac{\mu_I}{4}\hbar$ and secondly the infinite sum over stability angles.

Obtaining the energy spectrum from (1.6) is relatively straightforward since for a system to be semiclassically integrable requires that $[\hat{S}_i, \hat{S}_j] = O(\hbar^3)$ and so the energy eigenvalues are given to leading order in $\hbar$ simply by evaluating the classical energy $E_{\rm cl}[S_1, \ldots, S_{g+1}]$ on the eigenvalues of the action variables (1.6). As we show in section 5.1 this can be expanded to order $O(\hbar)$, expressing the result as a sum of the tree level term $E_{\rm cl}[N_1\hbar, \ldots, N_{g+1}\hbar]$ and the 1-loop correction involving the sum over stability angles. But moreover, in section 5.1 we also show, using the result of section 3.3, that the sum of the tree level term and 1-loop correction term can be succinctly rewritten in a compact form that captures the complete result at 1-loop in a unified way. Indeed, we show that the energy spectrum can be formally obtained by evaluating the classical energy of an *infinite*-gap solution with all its infinite filling fractions quantised to half-integer multiples of $\hbar$, namely

$$E = E_{\rm cl}\left[\left(N_1 + \frac{1}{2} \right) \hbar, \ldots \right].$$

This result is to be interpreted as a limit of expressions where a finite but *arbitrary* number of first entries are of order $O(1)$ corresponding to the tree level order and the remaining infinite number of entries encode the stability angle contribution to the 1-loop corrections of order $O(\hbar)$, see (5.5).

[1]Here and in the remainder of the paper, in the string theory context we will always let $\hbar = \frac{1}{\sqrt{\lambda}}$.

The paper is organised as follows: in section 2 we review some basic features of semiclassical quantisation for finite-dimensional systems. In particular we remind the reader how operator ordering enters in the semiclassical regime: in the language of pseudo-differential operators (appendix A) the different operator orderings are encoded in the subprincipal symbol [28]. We also sketch the derivation of the Bohr-Sommerfeld quantisation conditions [28, 22, 23]. In section 3 we look back at the general construction of finite-gap strings [13, 14] and derive the whole hierarchy of commuting flows. That is, we show how the integrable equations of motion for the embedding of the string in $\mathbb{R} \times S^3$ are part of an infinite hierarchy of higher integrable equations corresponding to the infinite set of conserved charges of the string, as is usual in any integrable system. In section 4 we discuss perturbations of a generic finite-gap string through the pinching of a-cycles. This leads to a general formula for the non-zero stability angles of a generic finite-gap string. Using this result, in section 5 we come back to the issue of semiclassical quantisation of finite-gap strings and apply the formalism of section 2 to obtain the semiclassical spectrum of the string. Some appendices elaborate on the discussion in each section.

2. Semiclassical approximation generalities

Consider a classical Hamiltonian system described by a $2n$ dimensional phase-space T^*X with Hamiltonian $H : T^*X \to \mathbb{R}$. Given $E \in \mathbb{R}$ we can consider the codimension one energy level set $\Sigma_E \equiv H^{-1}(E) \subset T^*X$. Assume also that we have a desired quantisation of the system, that is, we have a self-adjoint operator $\hat{H}$ acting on $L^2(X)$ whose principal symbol is the classical Hamiltonian H. If $H^{-1}([E-\epsilon, E+\epsilon])$ is compact then the eigenvalues of $\hat{H}$ in the range $[E-\epsilon, E+\epsilon]$ will be discrete since the corresponding eigenfunctions are localised around this compact set. The goal of semi-classical quantisation is to obtain the spectrum of $\hat{H}$ in $[E-\epsilon, E+\epsilon]$ to leading order in $\hbar$. One approach is to describe the spectrum using what are known as trace formulae, the basic idea being to encode the spectrum in terms of a single function $n(E) \equiv \sum_{j=0}^{\infty} \delta(E - E_j^\hbar) = \text{tr}\ \delta(E - \hat{H})$ where $E_j^\hbar$ denote the eigenvalues of $\hat{H}$ and which can be rewritten as

$$n(E) = \text{Re}\frac{1}{\pi\hbar}\int_0^\infty dt\ \text{tr}\, e^{\frac{i}{\hbar}(E-\hat{H})t} = \text{Re}\frac{1}{\pi\hbar}\int_0^\infty dt\ e^{\frac{iEt}{\hbar}} \int_{\substack{\text{p.o. } \gamma \\ \text{period } t}} [d\gamma] e^{-\frac{i}{\hbar}\int_\gamma \mathcal{L}}. \tag{2.1}$$

In the semiclassical limit $\hbar \to 0$ we perform a stationary phase approximation of the integral on the right hand side in order to obtain a semiclassical estimate of the spectrum $\{E_j^\hbar\}$ of $\hat{H}$. The presence of the trace means that dominating contributions come from periodic orbits of the classical system. This is a general feature of semiclassical trace formulae which relate *analytic* data of the operator $\hat{H}$ (namely its eigenvalues) to *geometric* data of the corresponding classical Hamiltonian H (namely its periodic orbits). This is one advantage of trace formulae over other semiclassical quantisation methods in that they elucidate the relation between the semiclassical spectrum and the classical periodic orbits.

On the downside however, despite the geometrical appeal of the path integral approach to semiclassical quantisation, it is hard to discuss the issues of operator ordering within this framework. Indeed, thinking in terms of phase-space path integrals, since everything in the

JHEP06(2008)086

integrand itself is classical, any information about quantum ordering is neatly tucked away in the definition of the regularisation used in the phase-space path integral measure $[d\gamma]$. The standard choice of discretisation of the path integral measure involves the mid-point prescription which corresponds to the Weyl-ordering prescription in the operator formalism. In particular the quantum Hamiltonian is the Weyl-ordered classical Hamiltonian, i.e. $\hat{H} = \mathrm{Op}^W_\hbar(H)$. In order to deal with operator ordering issues, it is therefore more convenient to work directly with operators.

A convenient operator formalism for discussing semi-classical quantisation involves pseudo-differential operators (referred to as ΨDOs for short). We refer to appendix A for a very brief introduction to ΨDOs and their relevance for treating semiclassical quantisation. The basic idea of this approach is to associate with any operator $\hat{f}$ not a single function on T^*X, which cannot by itself encode all the information about operator ordering in $\hat{f}$, but a family of functions $f_\hbar \in C(T^*X)$ called *symbols*. The leading function f_0 is exactly the classical function corresponding to $\hat{f}$, whereas all the subleading functions encode the operator ordering in $\hat{f}$. So instead of working with operators one can work directly with their respective symbols. Moreover, in the semiclassical approximation one only needs to deal with the first two symbols of an operator, known as the principal symbol (i.e. the classical function) and the subprincipal symbol. We will turn to the formalism of ΨDOs and the issue of operator ordering in an integrable system in section 2.1. In section 2.2 we will show how the Bohr-Sommerfeld quantisation conditions are modified by the presence of a subprincipal symbol which reflects a choice of ordering.

But first, to get an intuitive idea of how operator ordering ambiguities arise even at the semiclassical level to affect the quantisation conditions, it is instructive to consider the simple example of the harmonic oscillator for which the leading order quantisation is exact. The classical harmonic oscillator Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, and the action variable of the closed path of energy E is given by

$$I = \frac{1}{2\pi}\oint_{H=E} p dx = \frac{E}{\omega}.$$

By promoting the variables x, p to operators $\hat{x}, \hat{p}$ there is only one reasonable choice of ordering in the Hamiltonian, namely the Weyl-ordered Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. The spectrum of such an operator is well known to be $E_n = \left(n + \frac{1}{2}\right)\hbar\omega, n \in \mathbb{N}$ so that the spectrum of the Weyl-ordered action variable $\hat{I} = \frac{1}{\omega}\hat{H}$ is simply given by the standard Bohr-Sommerfeld quantisation condition,

$$\mathrm{Spec}\ (\hat{I}) \subset \left(\mathbb{Z} + \frac{1}{2}\right)\hbar,$$

where the index of $\frac{1}{2}$ by which the spectrum is shifted from $\hbar\mathbb{Z}$ is known as the Maslov index in the context of Bohr-Sommerfeld quantisation. Now since we are given at the outset only the classical Hamiltonian, we could always choose to quantise it with a more perverse choice of ordering. For instance, if we rewrite the classical Hamiltonian as $H = \omega a a^*$ where $a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{2m}\right)$ and after promoting everything to operators request that in the quantum Hamiltonian the $\hat{a}$ sits to the right of the $\hat{a}^\dagger$ then we obtain the normal-ordered Hamiltonian $:\hat{H}: = \omega\hbar\hat{a}^\dagger\hat{a}$, where $[\hat{a}, \hat{a}^\dagger] = 1$. The corresponding normal-ordered

action operator is given by $:\hat{I}: = \hbar\hat{a}^\dagger\hat{a}$ whose spectrum is easily seen to consists of integer multiples of $\hbar$,

$$\text{Spec}\,(:\hat{I}:) \subset \mathbb{Z}\hbar.$$

We observe that the Maslov index is precisely cancelled by the shift from Weyl-ordering to normal-ordering. Even though in the case of the harmonic oscillator we know that the correct physical quantisation of H is the Weyl-ordered one $\hat{H}$ we would like to stress that in general the choice of operator ordering in the quantisation of the action or Hamiltonian may not be as obvious and their spectrum may observe a shift from the standard Bohr-Sommerfeld spectrum $\left(\mathbb{Z} + \frac{\mu}{4}\right)\hbar$, where $\mu \in \mathbb{Z}_4$ is the Maslov index.

2.1 Operator ordering issues

As explained in appendix A, one can keep track of operator orderings in the language of pseudo-differential operators by retaining subleading terms beyond the principal symbol in the full Weyl symbol of an operator. In most applications of the theory of ΨDOs the quantities of interest are specified as ΨDOs at the outset so that their full Weyl symbol is known. In the present case however we start from a classical system specified by its phase-space (T^*X, ω) and the set of classical observables of interest are $F_1, \ldots, F_n, H$. Quantising this classical system requires an operator ordering prescription for obtaining operators from the corresponding classical observables. At the semiclassical level this boils down to the specification of an extra function, the subprincipal symbol, for each classical observable. Specifically, given a classical observable $f_0 \in C(T^*X)$, we construct

$$\hat{f} = \text{Op}^W_\hbar(f_0 + f_1\hbar),$$

where the presence of the subprincipal symbol $f_1 \in C(T^*X)$ reflects the operator ordering ambiguities already manifesting themselves at the semiclassical level. Every possible choice of a function $f_1 \in C(T^*X)$ corresponds to a different prescription for the operator ordering in $\hat{f}$ at order $O(\hbar)$. The principal symbol $f_0 = \sigma^W_0(\hat{f})$ is the corresponding classical observable.

Recall the definition of an integrable system, which roughly speaking is one which possesses the maximum possible number of independent integrals of motion. Specifically, a Hamiltonian system (T^*X, H) is said to be classically integrable if there exists n functions $F_1, \ldots, F_n \in C(T^*X)$ such that

(1′) $dF_1 \wedge \ldots \wedge dF_n \neq 0$ almost everywhere,

(2′) $\{F_i, F_j\} = 0$, $\forall i, j = 1, \ldots, n$,

(3′) $H = H(F_1, \ldots, F_n)$.

Conditions (2′) and (3′) together imply that the F_i are in fact integrals of motion, $X_H F_i = 0$. In other words, T^*X admits a torus action with moment map

$$\boldsymbol{F} \equiv (F_1, \ldots, F_n) : T^*X \to \mathbb{R}^n.$$

At regular values f of $\boldsymbol{F}$, the level sets $\boldsymbol{F}^{-1}(f)$ define n-torii (in the compact case) and foliate the phase-space T^*X; namely $\mathbb{T}^n \hookrightarrow T^*X \xrightarrow{\boldsymbol{F}} \mathbb{R}^n$. This foliation allows one to define canonical action-angle coordinates with the action variables $\{I_i\}_{i=1}^n$ parametrising the base $\mathbb{R}^n$ and the conjugate angle variables $\{\theta_i\}_{i=1}^n$, each taking values in $[0, 2\pi]$, parametrising the independent cycles of the torus $\mathbb{T}^n$.

We will say that a ΨDO $\hat{H}$ is *semiclassically integrable* if there exists n ΨDOs $\hat{F}_1, \ldots, \hat{F}_n$ with principal symbols $F_i = \sigma_0^W(\hat{F}_i)$ such that

(1) $dF_1 \wedge \ldots \wedge dF_n \neq 0$ almost everywhere,

(2) $[\hat{F}_i, \hat{F}_j] = O(\hbar^3),\ \forall i, j = 1, \ldots, n,$

(3) $\hat{H} = H(\hat{F}_1, \ldots, \hat{F}_n) + O(\hbar^2)$ for some function H.

Notice that we only require commutativity modulo $O(\hbar^3)$ in property (2); it guarantees in particular that the operator $H(\hat{F}_1, \ldots, \hat{F}_n)$ in (3) is free of operator ordering ambiguities certainly up to $O(\hbar^3)$, so that property (3) makes sense. Property (2) is to be contrasted with the definition of full quantum integrability which requires exact commutativity $[\hat{F}_i, \hat{F}_j] = 0$. Now since $\sigma_0^W([\hat{F}_i, \hat{F}_j]) = -i\hbar\{F_i, F_j\}$ (see appendix A) and $\sigma_0^W(\hat{H}) = H(F_1, \ldots, F_n)$, it follows that the principal symbols $F_i = \sigma_0^W(\hat{F}_i)$ satisfy all three properties (1′)-(3′) above for a classically integrable system with Hamiltonian $H = \sigma_0^W(\hat{H})$. This means that any semiclassically integrable system exhibits at leading order the full geometric structure of the underlying classically integrable system given by its principal symbols. In particular, the level set $\Lambda_f \equiv \boldsymbol{F}^{-1}(f)$ of the moment map $\boldsymbol{F} = (F_1, \ldots, F_n) : T^*X \to \mathbb{R}^n$ is a Lagrangian n-torus and foliates phase-space T^*X as we let f vary.

But the notion of semiclassical integrability contains more information than that of its underlying classical integrable structure [28, 29]. Property (1) only contributes at leading order since it is a statement about the principal symbols F_i alone, whereas property (2) at $O(\hbar^2)$ yields an equation for the subprincipal symbols $F_i^s = \sigma_{\text{sub}}^W(\hat{F}_i)$ of the $\hat{F}_i$ (see appendix A)

$$0 = \frac{i}{\hbar}\sigma_{\text{sub}}^W([\hat{F}_i, \hat{F}_j]) = \left\{F_i, F_j^s\right\} + \left\{F_i^s, F_j\right\}. \tag{2.2}$$

It is possible to interpret these equations geometrically so as to supplement the geometrical structure already laid out by the principal symbols with further geometrical data. For this we define the *subprincipal form* κ on Λ_f by defining its action on the basis vectors X_{F_i} at any point of Λ_f through [28]

$$\kappa(X_{F_i}) = -F_i^s, \quad i = 1, \ldots, n. \tag{2.3}$$

It then follows immediately from (2.2) that κ is closed since

$$\begin{aligned} d\kappa(X_{F_i}, X_{F_j}) &= X_{F_i}\kappa(X_{F_j}) - X_{F_j}\kappa(X_{F_i}) - \kappa([X_{F_i}, X_{F_j}]) \\ &= -X_{F_i}F_j^s + X_{F_j}F_i^s - \kappa(X_{\{F_i,F_j\}}) = -\{F_i, F_j^s\} + \{F_j, F_i^s\} = 0. \end{aligned}$$

Hence the operator ordering in the $\hat{F}_i$ can be accounted for at the semiclassical level by specifying a closed 1-form κ on the Liouville n-torus Λ_f. And in fact it is clear from (2.3) that every choice of a closed 1-form $\kappa \in \Omega^1(\Lambda_f)$ corresponds to a different choice of operator ordering in the definition of the $\hat{F}_i$.

2.2 Bohr-Sommerfeld conditions

We are interested in the joint spectrum of the $\hat{F}_i$ up to $O(\hbar)$ which requires solving the eingenvalue problem to that order

$$(\hat{F}_i - f_i)\psi = O(\hbar^2). \tag{2.4}$$

The Bohr-Sommerfeld conditions are conditions for the existence of a solution to these coupled pseudo-differential equations. Their rigourous derivation is rather involved but here we would just like to outline how the subprincipal symbol comes about in these conditions. To solve (2.4) locally one considers a local patch $V \subset \Lambda_f$ on which $\pi : T^*X \to X$ is a diffeomorphism and uses the WKB ansatz[2]

$$\psi_{\mathrm{WKB}} = e^{\frac{i}{\hbar}\phi_{-1}+\phi_0}\rho + O(\hbar)$$

on $U = \pi(V) \subset X$ where the nature of ρ will be specified shortly. If we let $\iota_{d\phi_{-1}} : U \hookrightarrow T^*X$ denote the 1-form $d\phi_{-1}$ viewed as a map then equation (2.4) implies to leading order in $\hbar$ that [20]

$$\mathrm{im}\ \iota_{d\phi_{-1}} = V \subset \Lambda_f, \tag{2.5}$$

or $\iota_{d\phi_{-1}} = \pi|_V^{-1}$. By a property of the tautological 1-form α, namely $d\phi_{-1} = \iota^*_{d\phi_{-1}}\alpha$, we then have [20]

$$d\pi|_V^*\phi_{-1} = \alpha, \tag{2.6}$$

in other words, $\pi|_V^*\phi_{-1}$ is a local solution to the classical integrability condition $\omega = d\alpha = 0$ on Λ_f. If ρ is a half-density[3] on $U \subset X$ then the subleading order of (2.4) can be written invariantly as (Theorem 11.11 p126 of [21])

$$\left(-i\mathcal{L}_{X_{F_i}} + F_i^s\right)\left(\pi|_V^* e^{i\phi_0}\rho\right) = 0.$$

Now provided the subprincipal symbols are real this equation implies on the one hand that $\pi|_V^*\rho$ is an invariant half-density on Λ_f, i.e. $\mathcal{L}_{X_{F_i}}\pi|_V^*\rho = 0$, and on the other hand that

$$d\pi|_V^*\phi_0 = \kappa, \tag{2.7}$$

which says that $\pi|_V^*\phi_0$ is a local solution to the subleading integrability condition that $d\kappa = 0$ on Λ_f. What one would like to do is patch up the local WKB solutions ψ_{WKB} defined on local neighbourhoods of Λ_f for which the projection $\pi : T^*X \to X$ is a diffeomorphism.

However, one runs into problems at caustic points where π is singular (see figure 6). A way around this problem was proposed by Maslov which allows one to define a solution to (2.4) which is localised and defined patchwise on Λ_f (near caustics one uses the "momentum" projection π_p of T^*X onto a typical fibre of T^*X instead of π). The single valuedness of this global solution requires its phase to be an integer multiple of 2π. The

[2]The WKB ansatz isn't actually restrictive since one can show that the space of solutions to (2.4) is one dimensional so that any other solution is proportional to the WKB solution.

[3]Since the product of two half-densities is a density of weight one there is a natural inner-product on half densities $\langle \rho_1, \rho_2 \rangle = \int_M \rho_1\rho_2$ which makes the completion into a Hilbert space.

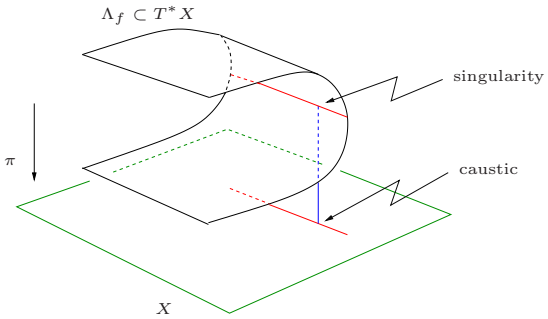


Figure 6: Caustics of the Lagrangian submanifold Λ_f

phase is essentially that of the local WKB solutions ψ_{WKB} introduced above but with additional Maslov index corrections (coming from the caustics) so that single valuedness conditions, known as the Bohr-Sommerfeld-Maslov conditions, read

$$\frac{1}{2\pi\hbar}\int_{\gamma_i}\alpha+\frac{1}{2\pi}\int_{\gamma_i}\kappa=N_i+\frac{\mu_{\gamma_i}}{4}+O(\hbar),\quad i=1,\ldots,n \tag{2.8}$$

where γ_i is a basis of $H_1(\Lambda_f,\mathbb{R})$ with Maslov indices $\mu_{\gamma_i}\in\mathbb{Z}_4$ and integers $N_i\in\mathbb{Z}$. Note in particular the presence of the subprincipal form κ which as we have argued is related to operator ordering ambiguities in going from a classically integrable system to its quantum (or just semiclassically) integrable counterpart. It has the effect of shifting the spectrum of the action variables similar to what happens in the case of the harmonic oscillator when we change quantisation, from Weyl to normal ordering say [24]. In the cases where all the operators are chosen to be Weyl ordered, in particular the $\hat{F}_i$, we have $\kappa=0$ and (2.9) reduces to the EBK quantisation conditions. In the remainder of the paper we shall make the assumption that the cohomology class $[\kappa]\in H^1(\Lambda_f)$ of the subprincipal form κ vanishes. The reason for this assumption is that the result is simpler to express in this case and moreover it agrees with the results of [16 – 18]. With this assumption, the Bohr-Sommerfeld-Maslov conditions simplify

$$\frac{1}{2\pi\hbar}\int_{\gamma_i}\alpha=N_i+\frac{\mu_{\gamma_i}}{4}+O(\hbar),\quad i=1,\ldots,n. \tag{2.9}$$

We stress that this assumption does *not* imply the choice of Weyl ordering since it only corresponds to setting the subprincipal symbol to zero, whereas Weyl ordering corresponds to setting all the lower order Weyl symbols to zero as well.

Now the derivation of the Bohr-Sommerfeld-Maslov conditions (2.8) or (2.9) essentially consisted in quantising a Lagrangian n-torus Λ_f by constructing a wave-function localised around it. However, even though the level set $\Lambda_f\equiv\boldsymbol{F}^{-1}(f)$ is indeed a Lagrangian n-torus for almost every value of the integrals of motion $f_1,\ldots,f_n$ in an integrable system, there exists interesting level sets $\boldsymbol{F}^{-1}(f)$ in phase-space where this is not the case. This happens at the (measure zero) set of critical values of the map $\boldsymbol{F}=(F_1,\ldots,F_n)$. Consider for instance the two-dimensional harmonic oscillator with different frequencies and

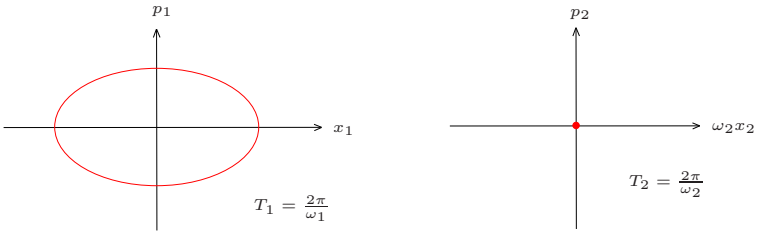


Figure 7: Periodic orbit with $H_2 = 0$ of energy $H = H_1 = E$.

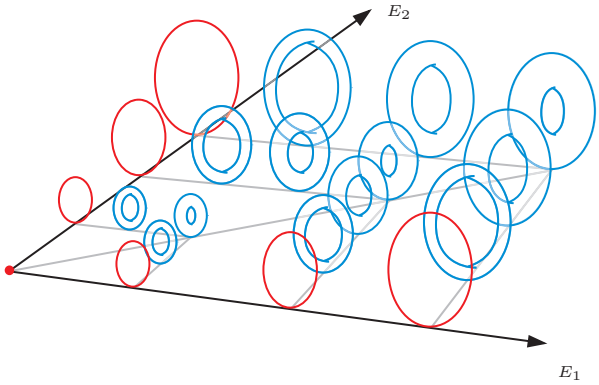


Figure 8: The phase-space of the two-dimensional harmonic oscillator.

Hamiltonian

$$H = \frac{p_1^2}{2} + \frac{1}{2}\omega_1^2 x_1^2 + \frac{p_2^2}{2} + \frac{1}{2}\omega_2^2 x_2^2 = H_1 + H_2, \tag{2.10}$$

whose integrals of motion are given by H_1, H_2. For non-zero values $E_1, E_2 \neq 0$ of H_1, H_2 the level sets $\boldsymbol{H}^{-1}(E_1, E_2)$ consists of two ellipses, in other words a Lagrangian 2-torus. However, if say $E_2 = 0$ the level set $\boldsymbol{H}^{-1}(E_1, 0)$ consists of just a single ellipse (figure 7). The same thing is true when $E_1 = 0$ and at the point where $E_1 = E_2 = 0$ the level set consists of just a single point. One can draw a picture of the phase-space in the region where $\mathcal{E} \equiv \{(E_1, E_2) : E_i \geq 0, i = 1, 2\}$ which is foliated by 2-torii in the interior of $\mathcal{E}$ but with the fibres over the boundary $\partial\mathcal{E} \setminus \{(0, 0)\}$ being ellipses and the fibre over the point $(0, 0)$ being just a single point, see figure 8. Note that the set of critical values $\partial\mathcal{E}$ is of measure zero. However, if we are interested in the semiclassical spectrum of the two-dimensional harmonic oscillator in the region near $\partial\mathcal{E}$ then a modification of the Bohr-Sommerfeld-Maslov quantisation conditions (2.9) is required so that it applies to isotropic p-torii which are the level sets of a limited number $p < n$ of integrals of motion $F_1, \ldots, F_p$.

It was pointed out by Voros [22, 23] that the Bohr-Sommerfeld conditions (2.9) for the apparently more restrictive case of an integrable system may be used to obtain the Bohr-Sommerfeld conditions in all other intermediate cases, namely the partially integrable one (with $p < n$ integrals of motion) and even the non-degenerate case $p = 1$ (where H is the only integral). If the system has p independent observables $\boldsymbol{F} = (F_1, \ldots, F_p)$ in involution (with $H = H(\boldsymbol{F})$), then on each codimension p level set $\Sigma_f = \boldsymbol{F}^{-1}(f)$ the system has a

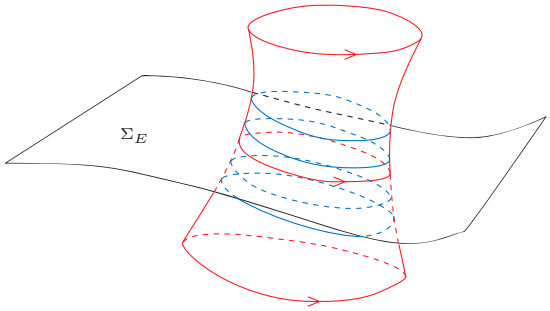


Figure 9: Bohr-Sommerfeld semi-classical spectrum: the discrete set of periodic orbits $\gamma(E^\hbar_{j_a})$ shown in blue have energies $E^\hbar_{j_a}$ approximating the eigenvalues of $\hat{H}$ to $O(\hbar^2)$.

p-torus $\Lambda_f \subset \Sigma_f$ generated by the vector fields X_{F_i}. Each of these p-torii is surrounded by an n-torus of the linearised system to which the Bohr-Sommerfeld-Maslov conditions (2.9) may be applied. This results in a set of Bohr-Sommerfeld conditions for the cycles on the p-torus which include stability angles for the small fluctuations in the directions transverse to this p-torus. The derivation of these Bohr-Sommerfeld conditions from those in the integrable case (2.9) are a bit lengthy but the derivation in the more general case $1 < p < n$ is conceptually the same as the $p = 1$ case [22, 23]. For completeness and to explain the appearance of the stability angles (which are related to the eigenvalues of the Poincaré map) we repeat the details of the derivation of [22, 23] in appendix B. The result (B.1) is the following quantisation condition for the *isolated* orbit γ [22, 23]

$$\int_\gamma \alpha = \left[2\pi\left(N + \frac{\mu_\gamma}{4}\right) + \sum_{\alpha=2}^{n}\left(n_\alpha + \frac{1}{2}\right)\nu_\alpha\right]\hbar + O(\hbar^2), \tag{2.11}$$

where $N \in \mathbb{Z}$ and $n_\alpha \in \mathbb{N}$. Since the periodic orbit $\gamma \subset \Sigma_E$ in fact belongs to a continuous 1-parameter family $\gamma(E)$ of periodic orbits parametrised by the energy E ('cylinder theorem'), what the condition (2.11) does is pick out a discrete set of periodic orbits $\gamma(E^\hbar_j)$, in a neighbourhood of Σ_E, whose energies $E^\hbar_j$ approximate eigenvalues of $\hat{H}$ to leading order in $\hbar$, see figure 9. The condition depends on the *stability angles* $\nu_\alpha \in \mathbb{R}$ (defined via the Poincaré map, see appendix B) of the stable isolated orbit γ and is valid only in the approximation where $0 < n_\alpha \ll |N|$ which is required for the linear approximation (used in deriving these condition) to hold.

The more general case of a system which has p independent observables $F_1, \ldots, F_p$ in involution (with $H = H(F_1, \ldots, F_p)$), where p lies in the range $1 < p < n$ is a straightforward generalisation. In this case we get a set of p quantisation conditions, one for each cycle $\gamma_k, k = 1, \ldots, p$ on the p-torus [22, 23],

$$\int_{\gamma_k} \alpha = \left[2\pi\left(N_k + \frac{\mu_{\gamma_k}}{4}\right) + \sum_{\alpha=p+1}^{n}\left(n^k_\alpha + \frac{1}{2}\right)\nu^k_\alpha\right]\hbar + O(\hbar^2). \tag{2.12}$$

This time there are p conditions on the p parameters $f_1, \ldots, f_p$ of the codimension p level sets $\Sigma_f = \boldsymbol{F}^{-1}(f)$.

JHEP06(2008)086

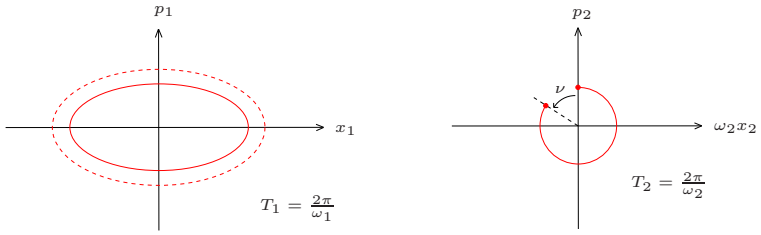


Figure 10: Perturbed trajectory of energy $H = H_1 + H_2 = E$.

To illustrate the use of the modified Bohr-Sommerfeld conditions (2.11) for an isolated orbit let us go back to the case of the two-dimensional harmonic oscillators (2.10). This system is obviously integrable and the exact spectrum of H is given by

$$E_{n_1,n_2} = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2.$$

However, suppose for the sake of argument that we can only solve classically for the Hamiltonian H_1 and wish to obtain the spectrum of $H = H_1 + H_2$ by perturbation as describe above. Then consider a particular motion of the Hamiltonian H_1 of total energy $H_1 = E$, through the point $(p_1, x_1, p_2, x_2) = (p_0, 0, 0, 0)$ say, see figure 7. This defines a 1-parameter family of periodic orbits parametrised by their energy $H = H_1 = E$. It is clear that the (p_2, x_2)-plane gives a Poincaré section of the orbit through the point $(p_0, 0, 0, 0)$ since all orbits of H_1 have the same period $T_1 = \frac{2\pi}{\omega_1}$. The prescription for determining the stability angles of this orbit is to consider small perturbations around it within the same energy level $H = E$. If the periods of the two harmonic oscillators are different, $T_1 \neq T_2$, then after a length of time T_1, the motion in the (p_2, x_2)-plane does not close and there is a deficit angle of $\nu = \omega_2 \cdot T_1$, see figure 10. The tower of energy levels corresponding to the periodic motion in figure 7 is therefore given by the Bohr-Sommerfeld condition (2.11) which in this case reads

$$I_1 = \left[\left(n_1 + \frac{1}{2}\right) + \left(n_2 + \frac{1}{2}\right)\frac{\nu}{2\pi}\right]\hbar + O(\hbar^2)$$

and hence $E_{n_1,n_2} = \omega_1 \cdot I_1 = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2 + O(\hbar^2)$ so that the Bohr-Sommerfeld condition is actually exact to first order in $\hbar$ on the harmonic oscillator.

3. The string hierarchy

In the general theory of finite-gap integration [30 – 36] the reconstruction of a solution requires an algebraic curve Σ, which specifies the integrals of motion, as well as a divisor (i.e. a finite set of points) $\hat{\gamma}_0$ on Σ which specifies the initial conditions for the dynamics. However, since the system is integrable it possesses an infinite number of integrals of motion H_i, each one generating a different Hamiltonian flow on phase-space in the usual sense through the Hamilton equations $\partial_{t_i} f = \{H_i, f\}$. The dynamics of the divisor $\hat{\gamma}(t_i)$ (with initial condition $\hat{\gamma}(0) = \hat{\gamma}_0$) on Σ with respect to some time t_i is then determined by

the corresponding Hamiltonian H_i. In fact, in this setup there is a natural correspondence between Hamiltonian flows and meromorphic differentials on the algebraic curve Σ. This is well known for instance in the case of the worldsheet coordinate σ which couples to the quasi-momentum dp. Indeed, it was shown in [13] that the worldsheet coordinates (σ, τ) enter the finite-gap solution only through the meromorphic differential

$$d\mathcal{Q} = \frac{1}{2\pi}\left(\sigma dp + \tau dq\right),$$

so that the coordinates σ and τ are said to "couple" respectively to the quasi-momentum dp and the quasi-energy dq. The aim of this section is to similarly identify the dynamics corresponding to all the higher conserved charges within the finite-gap language in terms of meromorphic differentials on the underlying curve Σ.

The reason for doing this is the following. As we have already mentioned in the introduction and will recall again in section 5, a finite-gap solution can be understood as an injective map from a finite dimensional phase-space to the full infinite dimensional phase-space of the theory [14, 32]. The Liouville torus of the finite dimensional phase-space in question is the (generalised) Jacobian $J(\Sigma, \infty^{\pm})$ of the algebraic curve Σ which is a $(g+1)$-torus. We will show that the divisor moves linearly on $J(\Sigma, \infty^{\pm})$ with respect to all the higher flows. But since $J(\Sigma, \infty^{\pm})$ is $g+1$ dimensional one can use $g+1$ independent such flows to parameterise it. This will give a nice coordinate system on the Jacobian which will be useful when we come to consider perturbations of this $(g+1)$-torus in section 4 for computing stability angles. In particular, the angle variables φ_I which will couple to the quasi-actions $dq^{(I)}$ defined in section 3.3 will parameterise $g+1$ independent cycles $\mathcal{C}_I \equiv \{\varphi_I \in [0, 2\pi)\}$ on the Jacobian along which the Poincaré maps will be defined. Many of the techniques used in this section can be found in the book [31].

3.1 Higher times and zero-curvature

If one can rewrite the equations of motion of an integrable two-dimensional field theory in the form of a zero-curvature equation for a one-parameter family of 1-forms $J(x)$, namely

$$dJ(x) - J(x) \wedge J(x) = 0, \tag{3.1}$$

then this leads straight away to the construction of an infinite set of conserved charges by considering the parallel transporter $\Omega(x)$ with connection $J(x)$ along a loop winding once around the worldsheet. The flatness of $J(x)$ immediately yields

$$\partial_\sigma \mathrm{tr}\ \Omega(x)^n = \partial_\tau \mathrm{tr}\ \Omega(x)^n = 0, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Moreover, as was shown in [14] the invariants $\mathrm{tr}\ \Omega(x)^n$ are in involution with respect to the Poisson bracket

$$\left\{\mathrm{tr}\ \Omega(x)^n, \mathrm{tr}\ \Omega(x')^m\right\} = 0, \quad \forall n, m \in \mathbb{N}. \tag{3.3}$$

This condition contains (3.2) as a special case since the worldsheet energy $\mathcal{E}$ and momentum $\mathcal{P}$ are related to the leading order asymptotic of $\Omega(x)$ near $x = \pm 1$. In fact (3.3) is the statement of the invariance of $\mathrm{tr}\ \Omega(x')^m$ with respect to an infinite family of higher

flows generated by tr $\Omega(x)^n$. We will now show that the Hamilton equations of motion corresponding to these higher conserved charges tr $\Omega(x)^n$ also take the form of a zero-curvature condition.

Let us start by determining the evolution of the space component $J_1(x)$ of the lax connection under the higher flows, namely $\{\text{tr }\Omega(x)^n, J_1(x')\}$. For this we first obtain the following Poisson bracket with the transfer matrix $T(\sigma_1,\sigma_2,x) = P\overleftarrow{\exp}\int_{\sigma_2}^{\sigma_1} d\sigma J_1(\sigma,x)$,

$$\{T(\sigma_1,\sigma_2,x)\overset{\otimes}{,}J_1(\sigma_3,x')\} = \int_{\sigma_2}^{\sigma_1} d\sigma (T(\sigma_1,\sigma,x)\otimes\mathbf{1})\{J_1(\sigma,x)\overset{\otimes}{,}J_1(\sigma_3,x')\}(T(\sigma,\sigma_2,x)\otimes\mathbf{1}), \tag{3.4}$$

which requires the Poisson bracket $\{J_1\overset{\otimes}{,}J_1\}$ given in [14], first obtained by J.-M. Maillet in [19] in the context of the principal chiral model

$$\begin{aligned}\big\{J_1(\sigma,x)\overset{\otimes}{,}J_1(\sigma_3,x')\big\} = &\big[r(x,x'),J_1(\sigma,x)\otimes\mathbf{1}+\mathbf{1}\otimes J_1(\sigma_3,x')\big]\,\delta(\sigma-\sigma_3)\\ &-\big[s(x,x'),J_1(\sigma,x)\otimes\mathbf{1}-\mathbf{1}\otimes J_1(\sigma_3,x')\big]\,\delta(\sigma-\sigma_3)\\ &-2s(x,x')\delta'(\sigma-\sigma_3),\end{aligned} \tag{3.5}$$

where

$$r(x,x') = -\frac{2\pi}{\sqrt{\lambda}}\frac{x^2+x'^2-2x^2x'^2}{(x-x')(1-x^2)(1-x'^2)}\eta,\quad s(x,x') = -\frac{2\pi}{\sqrt{\lambda}}\frac{x+x'}{(1-x^2)(1-x'^2)}\eta. \tag{3.6}$$

Inserting (3.5) into (3.4), integrating by parts for the δ'-term and using identities like

$$\begin{cases}\dfrac{\partial T}{\partial\sigma_1}(\sigma_1,\sigma_2,x) = J_1(\sigma_1,x)T(\sigma_1,\sigma_2,x)\\ \dfrac{\partial T}{\partial\sigma_2}(\sigma_1,\sigma_2,x) = -T(\sigma_1,\sigma_2,x)J_1(\sigma_2,x),\end{cases} \tag{3.7}$$

yields

$$\begin{aligned}&\{T(\sigma_1,\sigma_2,x)\overset{\otimes}{,}J_1(\sigma_3,x')\}\\ &\quad= -2(\delta(\sigma_3-\sigma_1)-\delta(\sigma_3-\sigma_2))(T(\sigma_1,\sigma_3,x)\otimes\mathbf{1})s(x,x')(T(\sigma_3,\sigma_2,x)\otimes\mathbf{1})\\ &\qquad+\epsilon(\sigma_1-\sigma_2)\chi(\sigma_3;\sigma_1,\sigma_2)(T(\sigma_1,\sigma_3,x)\otimes\mathbf{1})\\ &\qquad\times[(r+s)(x,x'),J_1(\sigma_3,x)\otimes\mathbf{1}+\mathbf{1}\otimes J_1(\sigma_3,x')](T(\sigma_3,\sigma_2,x)\otimes\mathbf{1}),\end{aligned} \tag{3.8}$$

where $\epsilon(\sigma) = \text{sign}(\sigma)$ is the usual sign function and $\chi(\sigma;\sigma_1,\sigma_2)$ is the characteristic function of the interval between σ_1 and σ_2. If we are working on the circle, let $\sigma_1 = \sigma+2\pi, \sigma_2 = \sigma, \sigma_3 = \sigma'$ and identify the monodromy matrix as $\Omega(\sigma,x) = T(\sigma+2\pi,\sigma,x)$ then the previous equation reduces to

$$\begin{aligned}&\{\Omega(\sigma,x)\overset{\otimes}{,}J_1(\sigma',x')\}\\ &\quad= (T(\sigma+2\pi,\sigma',x)\otimes\mathbf{1})[(r+s)(x,x'),J_1(\sigma',x)\otimes\mathbf{1}+\mathbf{1}\otimes J_1(\sigma',x')](T(\sigma',\sigma,x)\otimes\mathbf{1}).\end{aligned}$$

Making use of (3.7) again, we can rewrite this as

$$\{\Omega(\sigma,x)\overset{\otimes}{,}J_1(\sigma',x')\} = \partial_{\sigma'}\mathcal{J}(\sigma,\sigma',x,x') + [\mathcal{J}(\sigma,\sigma',x,x'),\mathbf{1}\otimes J_1(\sigma',x')], \tag{3.9}$$

where

$$\mathcal{J}(\sigma,\sigma',x,x') = (T(\sigma+2\pi,\sigma',x)\otimes\mathbf{1})(r+s)(x,x')(T(\sigma',\sigma,x)\otimes\mathbf{1}).$$

Taking the trace over the first factor of the tensor product yields

$$\{\operatorname{tr}\Omega(\sigma,x), J_1(\sigma',x')\} = \partial_{\sigma'}\mathcal{J}(\sigma,\sigma',x,x') + [\mathcal{J}(\sigma,\sigma',x,x'), J_1(\sigma',x')], \tag{3.10}$$

where $\mathcal{J}(\sigma,\sigma',x,x') = \operatorname{tr}_1\left[(T(\sigma',\sigma,x)T(\sigma+2\pi,\sigma',x)\otimes\mathbf{1})(r+s)(x,x')\right]$. In fact, using the translation invariance of the transfer matrix T by 2π and the definition of $\Omega(x)$, we see that $\mathcal{J}(\sigma,\sigma',x,x')$ does not explicitly depend on σ and can be written more succinctly as

$$\mathcal{J}(\sigma',x,x') = \operatorname{tr}_1\left[(\Omega(\sigma',x)\otimes\mathbf{1})(r+s)(x,x')\right]. \tag{3.11}$$

If we interpret the Poisson bracket $\{\operatorname{tr}\Omega(\sigma,x), J_1(\sigma',x')\}$ in (3.10) as the "time" derivative of $J_1(\sigma',x')$ with respect to the time generated by the Hamiltonian $\operatorname{tr}\Omega(x)$ then (3.10) takes exactly the form of a zero-curvature equation. This indicates that (3.11) ought to be related to the Lax matrices corresponding to all the higher order flows generated by the Hamiltonians $\operatorname{tr}\Omega(x)$, just as J_0 and J_1 were the Lax matrices generating τ and σ respectively. In fact, as we will show, one should Taylor expand (3.10) and (3.11) around $x=\pm1$ thereby obtaining a discrete set of independent times $t_{n,\pm}$.

An important remark is in order at this stage: since we are really doing string theory in conformal static gauge by imposing the Virasoro constraints and static gauge fixing conditions, which constitute a set of second class constraints in the Hamiltonian formalism, one should take care in imposing them consistently. This means that we should define an appropriate Dirac Bracket corresponding to every Poisson bracket and write everything in terms of those. Once this is done, the Virasoro constraints and static gauge fixing conditions can then be imposed without worry at any level of the calculation. However, as we show in appendix C, for all the brackets of interest in the following, the Dirac and Poisson brackets are identical. Thus in the remainder of this section we shall denote brackets by $\{\cdot,\cdot\}$ without specifying whether they are Dirac brackets or Poisson brackets.

Let us first obtain the equations of motion for the monodromy matrix with respect to the Hamiltonian $\operatorname{tr}\Omega(x)$. Starting from the Poisson algebra of the monodromies [19, 14],

$$\begin{aligned}\left\{\Omega(x)\overset{\otimes}{,}\Omega(x')\right\} =& [r(x,x'),\Omega(x)\otimes\Omega(x')] \\ &+ (\Omega(x)\otimes\mathbf{1})\, s(x,x')\left(\mathbf{1}\otimes\Omega(x')\right) \\ &- \left(\mathbf{1}\otimes\Omega(x')\right) s(x,x')\,(\Omega(x)\otimes\mathbf{1}),\end{aligned} \tag{3.12}$$

and taking the trace over the first factor of the tensor product as above yields

$$\{\operatorname{tr}\Omega(x),\Omega(x')\} = [\mathcal{J}(x,x'),\Omega(x')]. \tag{3.13}$$

Once again, if we interpret the Poisson bracket $\{\operatorname{tr}\Omega(x),\Omega(x')\}$ as a time derivative, this last equation starts to take the form of the (σ,τ)-evolution equations

$$[\partial_\tau - J_0(x'),\Omega(x')] = 0, \quad [\partial_\sigma - J_1(x'),\Omega(x')] = 0. \tag{3.14}$$

The expression (3.11) for the Lax matrices can be simplified further. Using (3.6) the sum of the (r,s)-matrices entering in (3.11) is

$$(r+s)(x,x') = -\frac{2\pi}{\sqrt{\lambda}}\frac{2x^2}{(x-x')(1-x^2)}\eta.$$

Now by definition, $\eta = -t^a \otimes t^a$ where the $\mathfrak{su}(2)$ generator t^a is related to the Pauli matrices as $t^a = \frac{i}{\sqrt{2}}\sigma_a$. Thus (3.11) can be written as

$$\mathcal{J}(\sigma',x,x') = -\frac{\pi}{\sqrt{\lambda}}\frac{2x^2}{(x-x')(1-x^2)}\mathrm{tr}\left[\Omega(\sigma',x)\,\sigma_a\right]\sigma_a. \tag{3.15}$$

Now it is straightforward to show that for any matrix $A \in \mathrm{SL}(2,\mathbb{C})$ the following is true

$$V^{-1}\frac{\mathrm{tr}\,[A\sigma_a]\sigma_a}{\lambda_+ - \lambda_-}V = \sigma_3, \qquad \text{where} \quad V^{-1}AV = \mathrm{diag}\,(\lambda_+,\lambda_-),$$

i.e. V is the matrix of eigenvectors of A and $\lambda_\pm$ are the eigenvalues. Since the eigenvalues of $\Omega(\sigma',x)$ are $e^{\pm ip(x)}$ and its matrix of eigenvectors is $\Psi(x)$, this identity implies that the Lax matrix (3.15) corresponding to the Hamiltonian $\mathrm{tr}\,\Omega(x)$ can be simplified as

$$\mathrm{tr}\,\Omega(x) \quad \longleftrightarrow \quad \mathcal{J}(x,x') = \frac{4\pi i}{\sqrt{\lambda}}\frac{\sin p(x)}{1-1/x^2}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'}. \tag{3.16}$$

But since $\mathrm{tr}\,\Omega(x) = 2\cos p(x)$, it follows that the Lax matrix responsible for the flow of the Hamiltonian $p(x)$ is

$$p(x) \quad \longleftrightarrow \quad J(x,x') = -\frac{2\pi i}{\sqrt{\lambda}}\frac{x^2}{x^2-1}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'}. \tag{3.17}$$

Now we expand this around $x = \pm 1$ by extracting the Lax matrices associated with the Taylor coefficients of the quasi-momentum about $x = \pm 1$, namely

$$\mathrm{res}_{x=\pm 1}\,(x\mp 1)^{-n}p(x) \quad \longleftrightarrow \quad \tilde{J}_{n,\pm}(x') = \mathrm{res}_{x=\pm 1}\,(x\mp 1)^{-n}J(x,x'). \tag{3.18}$$

Using the straightforward identity for a rational matrix $M(x)$ with singularities at $x = \pm 1$

$$\mathrm{res}_{x=\pm 1}\,\frac{M(x)}{x-x'} = -\left(M(x')\right)_{\pm 1}, \tag{3.19}$$

where $(M(x'))_{\pm 1}$ denotes the pole part of $M(x')$ at $x' = \pm 1$, one can recast the Lax matrix (3.18) in the much more useful form

$$\tilde{J}_{n,\pm}(x') = \left(\frac{2\pi i}{\sqrt{\lambda}}\frac{x'^2}{x'^2-1}\frac{\Psi(x')\sigma_3\Psi(x')^{-1}}{(x'\mp 1)^n}\right)_{\pm 1}. \tag{3.20}$$

At this point we can also define the corresponding *hierarchy of times* $\tilde{t}_{n,\pm}$ as the times generated by the Hamiltonians $\mathrm{res}_{x=\pm 1}\,(x\mp 1)^{-n}p(x)$ in (3.18), namely we define

$$\partial_{\tilde{t}_{n,\pm}} = \left\{\mathrm{res}_{x=\pm 1}\,(x\mp 1)^{-n}p(x),\cdot\right\}. \tag{3.21}$$

Going back to equation (3.13), if we follow the prescription we just established to go from (3.16) to (3.18), namely of dividing through by $-2\sin p(x)$ and taking the residue at $x = \pm 1$ one readily finds the equation governing the evolution of the monodromy matrix under the hierarchy of times (3.21)

$$[\partial_{\tilde{t}_{n,\pm}} - \tilde{J}_{n,\pm}(x'), \Omega(x')] = 0, \tag{3.22}$$

which is exactly of the form (3.14). As an application of equation (3.20) for the hierarchy of Lax matrices we show that the first two of these matrices $\tilde{J}_{0,\pm}$ are related to the original Lax connection $J_\pm = J_0 \pm J_1$. Indeed, applying the asymptotics for the quasi-momentum $p(x) \sim_{x\to\pm 1} -\frac{\pi\kappa_\pm}{x\mp 1}$ to equation (3.18) with $n = 0$ we find

$$-\pi\kappa_\pm \quad \longleftrightarrow \quad \tilde{J}_{0,\pm}(x') = \pm\frac{\pi i}{\sqrt{\lambda}}\frac{\Psi(\pm 1)\sigma_3\Psi(\pm 1)^{-1}}{x' \mp 1}.$$

Now the components $J_\pm$ of the Lax connection are associated to $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$ translations which are in turn generated by $\mathcal{E} \pm \mathcal{P} = \frac{\sqrt{\lambda}}{2}\kappa_\pm^2$ and hence $J_\pm(x') = -\frac{\sqrt{\lambda}\kappa_\pm}{\pi}\tilde{J}_{0,\pm}(x')$ since [13]

$$\mathcal{E} \pm \mathcal{P} \quad \longleftrightarrow \quad -\frac{\sqrt{\lambda}\kappa_\pm}{\pi}\tilde{J}_{0,\pm}(x') = \frac{i\kappa_\pm}{1 \mp x'}\Psi(\pm 1)\sigma_3\Psi(\pm 1)^{-1} = J_\pm(x'). \tag{3.23}$$

Finally we derive the evolution equations for the Lax matrices (3.20) under the hierarchy of times (3.21) and show that they take the zero-curvature form. We follow an argument given in [31] for finite-dimensional systems which applies readily here. Writing the monodromy matrix as $\Omega(x') = \Psi(x')\,\mathrm{diag}(e^{ip(x)}, e^{-ip(x)})\Psi(x')^{-1}$, equation (3.22) implies that

$$\left[\Psi(x')^{-1}\left(\partial_{\tilde{t}_{n,\pm}}\Psi(x')\right) - \Psi(x')^{-1}\tilde{J}_{n,\pm}(x')\Psi(x'), \mathrm{diag}(e^{ip(x)}, e^{-ip(x)})\right] = 0. \tag{3.24}$$

But any 2×2 matrix commuting with a diagonal matrix must itself be diagonal, and therefore we may write

$$\partial_{\tilde{t}_{n,\pm}}\Psi(x') = \tilde{J}_{n,\pm}(x')\Psi(x') + \Psi(x')D(x'), \tag{3.25}$$

for some unknown diagonal 2×2 matrix $D(x')$. Let us denote the multi-indices labelling the hierarchy, such as $(n, +)$, using capital letters, e.g. $N = (n, s_n)$ where $n \in \mathbb{N}$ and $s_n = \pm 1$. So let $N = (n, s_n)$ and $M = (m, s_m)$, then we have for $\tilde{J}_N(x') = \tilde{J}_{n,s_n}(x')$

$$\partial_{\tilde{t}_M}\tilde{J}_N(x') = \left[\tilde{J}_M(x'), \frac{2\pi i}{\sqrt{\lambda}}\frac{x'^2}{x'^2 - 1}\frac{\Psi(x')\sigma_3\Psi(x')^{-1}}{(x' - s_n)^n}\right]_{s_n}, \tag{3.26}$$

where we have made use of (3.25) and the subscript on the commutator means we take the pole part of the whole commutator at $x' = s_n$. Let us start by assuming that $s_n \neq s_m$, then $\tilde{J}_M(x')$ is regular at $x' = s_n$ and only the pole part at $x' = s_n$ of the second term in the commutator contributes which is just $\tilde{J}_N(x')$, so

$$\partial_{\tilde{t}_M}\tilde{J}_N(x') = [\tilde{J}_M(x'), \tilde{J}_N(x')]_{s_n},$$

and likewise we also have $\partial_{\tilde{t}_N}\tilde{J}_M(x') = [\tilde{J}_N(x'), \tilde{J}_M(x')]_{s_m}$. Since $[\tilde{J}_M(x'), \tilde{J}_N(x')]$ is rational with poles only at $x' = \pm 1$ and vanishes at $x' = \infty$ it can be written as a sum over its pole parts, namely

$$[\tilde{J}_M(x'), \tilde{J}_N(x')] = [\tilde{J}_M(x'), \tilde{J}_N(x')]_{+1} + [\tilde{J}_M(x'), \tilde{J}_N(x')]_{-1}.$$

But because $s_n \neq s_m$ we have $\{s_m, s_n\} = \{\pm 1\}$ and the zero-curvature condition (3.27) below follows. If instead we assume that $s_n = s_m$, then we have

$$\left[\tilde{J}_N(x') - \frac{2\pi i}{\sqrt{\lambda}}\frac{x'^2}{x'^2-1}\frac{\Psi(x')\sigma_3\Psi(x')^{-1}}{(x'-s_n)^n}, \tilde{J}_M(x') - \frac{2\pi i}{\sqrt{\lambda}}\frac{x'^2}{x'^2-1}\frac{\Psi(x')\sigma_3\Psi(x')^{-1}}{(x'-s_n)^m}\right]_{s_n} = 0$$

since both arguments in the commutator are regular at $x' = s_n = s_m$. The zero-curvature equation again readily follows from the above equation and (3.26), i.e.

$$\partial_{\tilde{t}_M}\tilde{J}_N(x') - \partial_{\tilde{t}_N}\tilde{J}_M(x') = [\tilde{J}_M(x'), \tilde{J}_N(x')]. \tag{3.27}$$

Let us give an alternative basis $J_{n,\pm}$ for the string hierarchy whose zeroth level $n = 0$ corresponds exactly to the Lax connection $J_\pm$. If we define $-\pi\kappa_{n,\pm} = \mathrm{res}_{x=\pm 1}(x \mp 1)^{-n}p(x)$ so that $\kappa_{0,\pm} = \kappa_\pm$ then we have the following correspondence between integral of motion and Lax connection

$$\frac{\sqrt{\lambda}}{2}\kappa_\pm\kappa_{n,\pm} \quad \longleftrightarrow \quad J_{n,\pm} = -\frac{\sqrt{\lambda}}{2\pi}\left(\kappa_\pm\tilde{J}_{n,\pm} + \kappa_{n,\pm}\tilde{J}_{0,\pm}\right).$$

In particular, from (3.23) we see that the zeroth level $n = 0$ of this hierarchy is precisely the Lax connection $J_\pm$ associated with $\mathcal{E} \pm \mathcal{P} = \frac{\sqrt{\lambda}}{2}\kappa_\pm^2$, so as desired $J_{0,\pm} = J_\pm$. It is straightforward to see by the linearity of the above expression for $J_{n,\pm}$ and the constancy of the integrals of motion $\kappa_{n,\pm}$ that the new hierarchy is also commuting, namely it also satisfies the zero-curvature equation (3.27), with $\partial_{t_{n,\pm}} = \{\frac{\sqrt{\lambda}}{2}\kappa_\pm\kappa_{n,\pm}, \cdot\}$

$$\partial_{t_M}J_N(x') - \partial_{t_N}J_M(x') = [J_M(x'), J_N(x')]. \tag{3.28a}$$

Likewise, equation (3.22) also goes through unaltered and reads

$$[\partial_{t_M} - J_M(x'), \Omega(x')] = 0. \tag{3.28b}$$

3.2 Baker-Akhiezer vector and linearization

Equations (3.28) express the fact that the operators $\partial_{t_M} - J_M(x')$ all commute among themselves as well as individually with the monodromy matrix $\Omega(x')$. This means they can all be simultaneously diagonalised and there exists a solution $\boldsymbol{\psi}(P')$ to the following equations, where $P' = (x', y') \in \Gamma$ and $\Gamma : \det(\Omega(x') - y'\mathbf{1}) = 0$ is the spectral curve,

$$\begin{cases} \big(\partial_{t_M} - J_M(x')\big)\boldsymbol{\psi}(P') = 0, & \forall M \\ (\Omega(x') - y')\boldsymbol{\psi}(P') = 0. \end{cases} \tag{3.29}$$

In this section it will be important to keep track of the explicit dependence of various functions on the hierarchy of times and so we will use the notation $\{t\}$ for the complete set of times $t_{0,\pm}, t_{1,\pm}, \ldots$ and write for instance $J_M(x', \{t\})$, $\Omega(x', \{t\})$ and $\boldsymbol{\psi}(P', \{t\})$.

The idea of finite-gap integration (see [13, 14] and references therein) is to identify the analytic properties of the vector $\boldsymbol{\psi}(P', \{t\})$ which specify it uniquely. To this aim we follow [14, 36] and introduce the normalised eigenvector $\boldsymbol{h}(P', \{t\})$ of $\Omega(x')$ which is normalised by the condition $\boldsymbol{\alpha} \cdot \boldsymbol{h} = 1$ where $\boldsymbol{\alpha} = (1, 1)$. Using this vector we can look for solutions to (3.29) in the form

$$\boldsymbol{\psi}(P', \{t\}) = \widehat{\Psi}(x', \{t\})\boldsymbol{h}(P', \{0\}), \tag{3.30}$$

where $\widehat{\Psi}(x', \{t\})$ is a formal matrix solution to $\big(\partial_{t_M} - J_M(x')\big)\widehat{\Psi}(x') = 0$, $\forall M$ so that (3.30) trivially satisfies $\big(\partial_{t_M} - J_M(x')\big)\boldsymbol{\psi}(P') = 0$, $\forall M$. If we fix the initial condition to be $\boldsymbol{\psi}(P', \{0\}) = \boldsymbol{h}(P', \{0\})$ so that $\widehat{\Psi}(x', \{0\}) = \mathbf{1}$ then by uniqueness of the solution with initial condition $\widehat{\Psi}(x', \{0\}) = \Omega(x', \{0\})$ it follows that $\widehat{\Psi}(x', \{t\})\Omega(x', \{0\}) = \Omega(x', \{t\})\widehat{\Psi}(x', \{t\})$ and therefore (3.30) is indeed also an eigenvector of $\Omega(x', \{t\})$.

We now analyse the analytic properties of the vector $\boldsymbol{\psi}(P', \{t\})$ in the form (3.30) by obtaining the analytic properties of $\widehat{\Psi}(x')$ and $\boldsymbol{h}(P', \{0\})$. First let us rewrite the hierarchy of Lax matrices in the more transparent form

$$J_{n,\pm}(x') = \big(\Psi(x')s_{n,\pm}(x')\sigma_3\Psi(x')^{-1}\big)_{\pm 1},$$

where the *singular parts* $s_{n,\pm}(x')$ are defined as

$$s_{n,\pm}(x') = \left(-i\frac{x'^2}{x'^2 - 1}\left(\kappa_{n,\pm} + \frac{\kappa_\pm}{(x' \mp 1)^n}\right)\right)_{\pm 1}. \tag{3.31}$$

In the particular case of the zeroth level Lax matrix $J_{0,\pm}(x')$ the singular parts are precisely those of the Lax connection $J_\pm$ as defined in [13, 14], namely $s_{0,\pm}(x') = \frac{i\kappa_\pm}{1 \mp x'}$. Because $J_{n,\pm}(x')$ only has poles at $x' = \pm 1$ it follows by Poincaré's theorem on holomorphic differential equations that $\widehat{\Psi}(x')$ is holomorphic outside $x' = \pm 1$. By studying the asymptotics of the equation for $\widehat{\Psi}(x')$, its behaviour near $x' = \pm 1$ is easily show to be

$$\widehat{\Psi}(x', \{t\})e^{-\sum_n s_{n,\pm}t_{n,\pm}\sigma_3} = O(1) \qquad \text{as } x \to \pm 1,$$

where $O(1)$ denotes a matrix holomorphic in a neighbourhood of $x' = \pm 1$. Moreover, using the fact that $J_{n,\pm}(\infty) = 0$ we observe that $\partial_{t_M}\widehat{\Psi}(\infty, \{t\}) = 0$, $\forall M$ and hence $\widehat{\Psi}(\infty, \{t\}) = \mathbf{1}$ by the choice of initial conditions. Turning to the normalised eigenvector $\boldsymbol{h}(P', \{t\})$, a standard analysis of its analytic behaviour reveals that it is meromorphic in P' and uniquely specified by the following condition

$$(h_1) \geq \hat{\gamma}(\{t\})^{-1}\infty^-, \quad h_1(\infty^+) = 1, \quad \text{and} \quad (h_2) \geq \hat{\gamma}(\{t\})^{-1}\infty^+, \quad h_1(\infty^-) = 1,$$

where the divisor $\hat{\gamma}(\{t\})$ of degree $g + 1$ is called the *dynamical divisor*. The analytic data gathered above for $\widehat{\Psi}(x')$ and $\boldsymbol{h}(P', \{0\})$ is sufficient to uniquely characterise the components of $\boldsymbol{\psi}(P', \{t\})$ as Baker-Akhiezer functions, namely

$$(\psi_1) \geq \hat{\gamma}_0^{-1}\infty^-, \quad \psi_1(\infty^+) = 1, \quad \text{and} \quad (\psi_2) \geq \hat{\gamma}_0^{-1}\infty^+, \quad \psi_2(\infty^-) = 1,$$
$$\text{with} \quad \begin{cases} \psi_i(x'^{\pm}, \{t\})e^{\mp\sum_n s_{n,+}t_{n,+}} = O(1), & \text{as } x' \to 1, \\ \psi_i(x'^{\pm}, \{t\})e^{\mp\sum_n s_{n,-}t_{n,-}} = O(1), & \text{as } x' \to -1, \end{cases}$$

where $\hat{\gamma}_0 = \hat{\gamma}(\{0\})$ is the initial divisor. Notice that the hierarchy of times enters linearly in the definition of the Baker-Akhiezer vector $\boldsymbol{\psi}(P', \{t\})$ through the essential singularity. This is a very general feature of finite-gap integration. When explicitly reconstructing the Baker-Akhiezer vector satisfying the above conditions in terms of Riemann θ-functions on Σ, the singular parts give rise to a unique normalised Abelian differential of the second kind $d\mathcal{Q}$ with poles at $x' = \pm 1$ of the prescribed form

$$d\mathcal{Q} = idS_{\pm}, \quad \text{as } x' \to \pm 1, \qquad \text{where } \begin{cases} S_+(x'^{\pm}, \{t\}) = \pm \sum_n s_{n,+}(x') t_{n,+}, \\ S_-(x'^{\pm}, \{t\}) = \pm \sum_n s_{n,-}(x') t_{n,-}. \end{cases}$$

All the time dependence of the Baker-Akhiezer vector, and hence of the solution, is encoded in this meromorphic differential $d\mathcal{Q}$ which is linear in the hierarchy of times. In fact, we can define a differential associated to each time of the hierarchy by writing

$$d\mathcal{Q} = \sum_n t_{n,+} d\Omega_{n,+} + \sum_n t_{n,-} d\Omega_{n,-} = \sum_N t_N d\Omega_N, \tag{3.32}$$

using the multi-index notation, where the normalised Abelian differentials of the second kind $d\Omega_{n,\pm}$ are defined uniquely by their respective behaviours at the points $x' = \pm 1$, namely

$$d\Omega_{n,+}(x'^{\pm}) = \pm i d s_{n,+}(x') \quad \text{as } x' \to +1, \qquad d\Omega_{n,-}(x'^{\pm}) = \pm i d s_{n,-}(x') \quad \text{as } x' \to -1.$$

This correspondence between times of the hierarchy and meromorphic differentials on Σ

$$t_{n,\pm} \mapsto d\Omega_{n,\pm}$$

is a very general feature of finite-gap integration. We say that the differential *couples* to the time for obvious reasons from (3.32). As we saw in the previous sections, every Hamlitonian corresponds to a Lax matrix which is responsible for generating the corresponding time in the Lax formalism. Here we see that every Hamiltonian also corresponds to a meromorphic differential on Σ responsible for generating the corresponding time in the finite-gap language. Notice the splitting between differentials singular at $x' = +1$ and those singular at $x' = -1$. These are related to left and right movers of the string. For instance, at the zeroth level $n = 0$ we have $\sigma^{\pm} \equiv \frac{\tau \pm \sigma}{2} = -t_{0,\pm}$ and $dq_{\pm} \equiv dq \pm dp = -2\pi d\Omega_{0,\pm}$, so in particular

$$t_{0,+} d\Omega_{0,+} + t_{0,-} d\Omega_{0,-} = \frac{1}{2\pi}(\sigma dp + \tau dq),$$

which is the usual $d\mathcal{Q}$ defined in [13 – 15] where all the higher times are set to zero.

In the next section we will be perturbing finite-gap solutions and so we give here the explicit formulae for the generic finite-gap solution in terms of Riemann θ-functions on Σ. Details can be found in [13 – 15]. Of particular interest for constructing the embedding g of the string in SU(2)

$$g = \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in \mathrm{SU}(2),$$

JHEP06(2008)086

is the dual Baker-Akhiezer vector which is defined relative to the conjugate divisor $\hat{\tau}\hat{\gamma}_0$ and opposite singular parts $-s_{n,\pm}$. The components are explicitly constructed as [14, 15]

$$Z_1 = C\widetilde{\psi}_1^+(0^+), \quad Z_2 = \frac{C}{\chi(\infty^-)^{\frac{1}{2}}}\widetilde{\psi}_2^+(0^+), \tag{3.33a}$$

where $C \in \mathbb{R}$ is a normalisation constant chosen such that $|Z_1|^2 + |Z_2|^2 = 1$ and $\chi(P)$ is a meromorphic function on Σ with divisor $(\chi) = \hat{\gamma}_0 \cdot \hat{\tau}\hat{\gamma}_0 \cdot B^{-1}$ and normalised by $\chi(\infty^+) = 1$ (B is the divisor of branch points of Σ). The components of the dual Baker-Akhiezer vector at 0^+ are explicitly given by

$$\widetilde{\psi}_1^+(0^+) = h_-(0^+)\frac{\theta\big(\boldsymbol{D};\Pi\big)\theta\big(2\pi\int_{\infty^+}^{0^+}\boldsymbol{\omega} - \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{D};\Pi\big)}{\theta\big(\int_{\boldsymbol{b}} d\mathcal{Q} + \boldsymbol{D};\Pi\big)\theta\big(2\pi\int_{\infty^+}^{0^+}\boldsymbol{\omega} - \boldsymbol{D};\Pi\big)}\exp\left(+\frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q} - \frac{i}{2}\int_{0^-}^{0^+} d\mathcal{Q}\right), \tag{3.33b}$$

$$\widetilde{\psi}_2^+(0^+) = h_+(0^+)\frac{\theta\big(\boldsymbol{D};\Pi\big)\theta\big(2\pi\int_{\infty^-}^{0^+}\boldsymbol{\omega} - \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{D};\Pi\big)}{\theta\big(\int_{\boldsymbol{b}} d\mathcal{Q} + \boldsymbol{D};\Pi\big)\theta\big(2\pi\int_{\infty^-}^{0^+}\boldsymbol{\omega} - \boldsymbol{D};\Pi\big)}\exp\left(-\frac{i}{2}\int_{\infty^-}^{\infty^+} d\mathcal{Q} - \frac{i}{2}\int_{0^-}^{0^+} d\mathcal{Q}\right). \tag{3.33c}$$

3.3 Quasi-actions

Remember that the Lax matrix in (3.17) is responsible for the flow of the Hamiltonian $\operatorname{tr}\Omega(x) = 2\cos p(x)$. Thus going back to the corresponding Hamilton equation in Lax form we can rewrite it as

$$2\pi i\left\{-\frac{\sqrt{\lambda}}{8\pi^2 i}\left(1 - \frac{1}{x^2}\right)p(x), J_1(x')\right\} = \left[\partial_\sigma - J_1(x'), \frac{\Psi(x)\frac{i}{2}\sigma_3\Psi(x)^{-1}}{x - x'}\right]. \tag{3.34}$$

Integrating this equation in x over the different a-cycles, and recalling that the action variables are defined as $S_i = -\frac{\sqrt{\lambda}}{8\pi^2 i}\int_{a_i}\left(1 - \frac{1}{x^2}\right)p(x)dx$ we find

$$\{S_i, J_1(x')\} = \left[\partial_\sigma - J_1(x'), \frac{1}{4\pi}\int_{a_i}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'}dx\right], \tag{3.35a}$$

and similarly integrating around the point $x = \infty$ and recalling that the global $\mathrm{SU}(2)_R$ charge is defined as $\frac{R}{2} = \frac{\sqrt{\lambda}}{8\pi^2 i}\oint_\infty\left(1 - \frac{1}{x^2}\right)p(x)dx$ we find

$$\frac{1}{2}\{R, J_1(x')\} = \left[\partial_\sigma - J_1(x'), -\frac{1}{4\pi}\int_{\infty}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'}dx\right], \tag{3.35b}$$

Equations (3.35) simply say that the Hamiltonian flow of the action variables S_i and R are generated by the following respective Lax matrices

$$\begin{aligned} S_i \quad &\longleftrightarrow \quad A_i(x') = \frac{1}{4\pi}\int_{a_i}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'}dx, \\ \frac{R}{2} \quad &\longleftrightarrow \quad -\frac{1}{4\pi}\oint_\infty\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x - x'}dx. \end{aligned} \tag{3.36}$$

Because any integral of motion can be expressed in terms of the action variables S_i, one ought to be able to use equation (3.36) to derive the Lax matrix for any other integral of motion. Indeed, for instance we know that

$$\begin{aligned}\delta\mathcal{P} &= \sum_{i=1}^{g}\left(\int_{b_i}\frac{dp}{2\pi}\right)\delta S_i + \left(\int_{\infty^-}^{\infty^+}\frac{dp}{2\pi}\right)\frac{1}{2}\delta R,\\ \delta\mathcal{E} &= \sum_{i=1}^{g}\left(\int_{b_i}\frac{dq}{2\pi}\right)\delta S_i + \left(\int_{\infty^-}^{\infty^+}\frac{dq}{2\pi}\right)\frac{1}{2}\delta R\end{aligned} \tag{3.37}$$

and so this means one can write

$$\{\mathcal{E}\pm\mathcal{P},\cdot\} = \sum_{i=1}^{g}\left(\int_{b_i}\frac{dq_\pm}{2\pi}\right)\{S_i,\cdot\} + \left(\int_{\infty^-}^{\infty^+}\frac{dq_\pm}{2\pi}\right)\frac{1}{2}\{R,\cdot\},$$

where $dq_\pm = dq \pm dp$. Making use of the Lax matrix for the action variables (3.36) and the fact that the differentials $dq_\pm$ are normalised as $\int_{a_i} dq_\pm = 0$ means we can write the Lax matrix for $\mathcal{E}\pm\mathcal{P}$ as follows

$$\begin{aligned}\mathcal{E}\pm\mathcal{P} \quad\longleftrightarrow\quad \frac{1}{4\pi}\sum_{i=1}^{g}\ &\left[\int_{a_i}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'}dx\int_{b_i}\frac{dq_\pm}{2\pi} - \int_{b_i}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'}dx\int_{a_i}\frac{dq_\pm}{2\pi}\right]\\ &-\frac{1}{4\pi}\oint_{\infty}\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'}dx\int_{\infty^-}^{\infty^+}\frac{dq_\pm}{2\pi}.\end{aligned}$$

Written in this form we can apply the Riemann bilinear identity to obtain

$$\mathcal{E}\pm\mathcal{P} \quad\longleftrightarrow\quad -i\Big(\mathrm{res}_{x=1}+\mathrm{res}_{x=-1}\Big)\frac{\Psi(x)\sigma_3\Psi(x)^{-1}}{x-x'}\frac{q_\pm(x)}{2\pi}dx, \tag{3.38}$$

where an overall factor of two came from the fact that we get equivalent contributions from both sheets, namely at $x^\pm = (+1)^\pm$ and $x^\pm = (-1)^\pm$. Note also importantly that there is no contribution from the apparent pole at $x = x'$. This is because $x = x'$ is not actually a pole of the Lax equation itself, as can be seen from (3.34) which is perfectly regular as x approaches x' since $[\partial_{t_{n,\pm}} - J_{n,\pm}(x'), \Psi(x')\sigma_3\Psi(x')^{-1}] = 0$ which follows from (3.25) and the trivial fact that diagonal matrices commute. An equation such as (3.38) relating an integral of motion to a Lax matrix should really always be understood as a relation between two ingredients of a Lax equation. To evaluate the residues in (3.38) we note that the Abelian integrals $q_\pm(x)$ have poles at $x = \pm 1$ with the following asymptotics

$$q_+(x) \sim_{x\to+1} -\frac{2\pi\kappa_+}{x-1}, \qquad q_-(x) \sim_{x\to-1} \frac{2\pi\kappa_-}{x+1}.$$

It follows now using the identity (3.19) that

$$\mathcal{E}\pm\mathcal{P} \quad\longleftrightarrow\quad \left(\Psi(x')i\sigma_3\Psi(x')^{-1}\frac{q_\pm(x')}{2\pi}\right)_{\pm1} = \frac{i\kappa_\pm}{1\mp x'}\Psi(\pm1)\sigma_3\Psi(\pm1)^{-1} = J_\pm(x'),$$

and we recover exactly the same expression as before (3.23). It is important to note that it was the multivaluedness of the Abelian integral $q_\pm(P) = \int^P dq_\pm$ (or equivalently the

fact that $dq_\pm$ had some non-trivial periods) which resulted in a non-zero answer for the corresponding Lax matrix. Indeed, the Lax matrix obtained by this argument clearly depends only on the cohomology class $[dq_\pm] \in H^1(\Sigma, \infty^\pm)$ of the Abelian differential $dq_\pm$ one starts off with on the singular algebraic curve $\Sigma/\{\infty^\pm\}$. One can see this explicitly from the equation preceding (3.38) or otherwise from (3.38) itself: suppose $dq_\pm, dq'_\pm$ are two representatives of the same cohomology class, then $dq_\pm - dq'_\pm = df$ is exact and the corresponding difference of the expressions in (3.38) is

$$-\frac{i}{2}\sum_{P=(\pm 1)^\pm} \mathrm{res}_P \frac{\Psi(P)\sigma_3\Psi(P)^{-1}}{x(P)-x'}\frac{f(P)}{2\pi}dx,$$

where $\Psi(P) = (\boldsymbol{h}(P), \boldsymbol{h}(\hat{\sigma}P))$. But this is the sum over the residues of a well defined meromorphic differential on $\Sigma/\{\infty^\pm\}$ (since $f(P)$ is single-valued) and so is zero.

One could use the same trick as above to compute more explicitly the Lax matrix for the action variables (3.36). To simplify the notation we first combine S_i and $\frac{1}{2}R$ into the $g+1$ filling fractions

$$S_I = -\frac{\sqrt{\lambda}}{8\pi^2 i}\int_{\mathcal{A}_I}\left(1-\frac{1}{x^2}\right)p(x)dx$$

where $\mathcal{A}_I$ is the cycle going around the I^{th} cut counterclockwise on the top sheet. They satisfy $\sum_{I=1}^{g+1} S_I = \frac{1}{2}(L-R)$ where L is the global $\mathrm{SU}(2)_L$ charge. So to apply the previous reasoning we could write

$$\delta S_I = \sum_{J=1}^{g+1}\delta_{IJ}\delta S_J.$$

For the same argument to follow through we must introduce Abelian differentials $dq^{(J)}$ of the second kind (so $dq^{(J)}$ has no residues) such that

$$\int_{\mathcal{A}_I} dq^{(J)} = 0, \qquad \int_{\mathcal{B}_I} dq^{(J)} = \delta_{IJ}, \tag{3.39}$$

where $\mathcal{B}_I$ is the contour going from ∞^+ to ∞^- through the I^{th} cut. Such differentials exist: consider $g+1$ independent differentials from the hierarchy, and call them $d\Omega_J$. Then the $(g+1)\times(g+1)$ matrix $A_{IJ} = \int_{\mathcal{B}_I} d\Omega_J$ is invertible, and so $dq^{(J)} = A^{-1}_{KJ}d\Omega_K$ have the desired property. Yet since the conditions (3.39) on the differentials $dq^{(J)}$ uniquely specify their cohomology class in $H^1(\Sigma, \infty^\pm)$, by the preceding remark they are also sufficient to uniquely fix the resulting Lax matrix. By the procedure of section 3.2 these Lax matrices yield unique normalised Abelian differentials which satisfy (3.39), which we still denote $dq^{(J)}$ by abuse of notation. Since the operations of constructing a Lax matrix from a given integral of motion and that of constructing an Abelian differential from a given Lax matrix are both linear, it follows that the equation for $\mathcal{E}$ in (3.37) translates into an equation in terms of differential forms on $\Sigma/\{\infty^\pm\}$. Rewrite this equation as

$$\delta\mathcal{E} = \sum_{i=1}^{g}\left(\int_{b_i}\frac{dq}{2\pi} - \int_{\infty^-}^{\infty^+}\frac{dq}{2\pi}\right)\delta S_i + \left(\int_{\infty^-}^{\infty^+}\frac{dq}{2\pi}\right)\delta\left(\frac{1}{2}R + \sum_{i=1}^{g} S_i\right) = \sum_{I=1}^{g+1}\left(\int_{\mathcal{B}_I}\frac{dq}{2\pi}\right)\delta S_I, \tag{3.40}$$

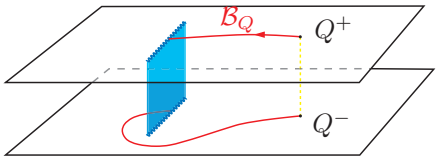


Figure 11: Definition of the cycle $\mathcal{B}_Q$ for a given singular point Q.

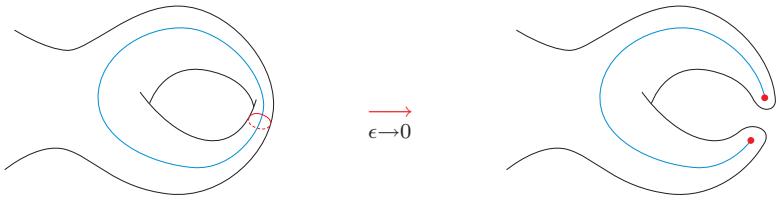


Figure 12: Pinching an a-cycle.

it follows that

$$dq = \sum_{I=1}^{g+1} \left(\int_{\mathcal{B}_I} dq \right) dq^{(I)}, \tag{3.41}$$

and in particular this leads to the following equation which will be important later

$$\int_{\mathcal{B}_Q} dq = \sum_{I=1}^{g+1} \left(\int_{\mathcal{B}_I} dq \right) \int_{\mathcal{B}_Q} dq^{(I)}$$

where $Q \in \Gamma$ is a singular point on Γ which is blown up on the desingularised curve Σ to two points $Q^\pm \in \Sigma$ and $\mathcal{B}_Q$ is a curve joining $Q^\pm$ having zero intersection number with any of the a- or b-cycles, see figure 11.

4. Perturbations of finite-gap strings

Given a finite-gap solution Z_i with underlying algebraic curve Σ of genus g, one can obtain its stability angles by considering nearby solutions $Z_i + \delta Z_i$ with algebraic curves Σ^ϵ of genus $g+1$. In other words, perturbations of a given finite-gap solution Z_i correspond to degenerations of a genus $g+1$ algebraic curve Σ^ϵ into the genus g curve Σ of the solution Z_i, see figure 12. Now since we are concerned with real finite-gap solutions, constructed from real algebraic curves Σ (see [13] for a discussion of reality conditions), the degeneration process in figure 12 describing the perturbation should respect this reality condition. This forces us to consider degenerations through the pinching of imaginary cycles, which we can choose to call the a-cycles as in [13]. We discuss the pinching of a-cycles in appendix D.

As discussed in section 3.2 the dependence of the general finite-gap solution on the hierarchy of times $\{t\}$ is entirely encoded in the normalised Abelian differential of the second kind $d\mathcal{Q} = \sum_N t_N d\Omega_N$ defined in (3.32) which enters the reconstruction formula as follows

$$Z_i = C_i \, \frac{\theta\big(2\pi \int_{P_i}^{0^+} \boldsymbol{\omega} - \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{D}; \Pi\big)}{\theta\big(\int_{\boldsymbol{b}} d\mathcal{Q} + \boldsymbol{D}; \Pi\big)} \, \exp\left(-i \int_{P_i}^{0^+} d\mathcal{Q} \right), \tag{4.1}$$

where $P_1 = \infty^+$ and $P_2 = \infty^-$. In this expression we have hidden all the time independent part into the overall constants C_i whose specific forms can be retrieved from the complete reconstruction formulae (3.33). A nearby solution $Z_i + \delta Z_i$ is constructed with the same formulae but from slightly deformed data (which includes a deformed curve Σ^ϵ)

$$Z_i + \delta Z_i = C_i^\epsilon \, \frac{\theta\big(2\pi \int_{P_i}^{0^+} \vec{\omega}^\epsilon - \int_{\vec{b}^\epsilon} d\mathcal{Q}^\epsilon - \vec{D}; \tilde{\Pi}^\epsilon\big)}{\theta\big(\int_{\vec{b}^\epsilon} d\mathcal{Q}^\epsilon + \vec{D}; \tilde{\Pi}^\epsilon\big)} \, \exp\left(-i \int_{P_i}^{0^+} d\mathcal{Q}^\epsilon\right). \tag{4.2}$$

The ingredients of the deformed solution are as follows. First of all, since the underlying curve Σ^ϵ has genus $g+1$, the arguments of the θ-functions for this curve are $(g+1)$-component vectors, namely

$$\vec{D} = \begin{pmatrix} D_0 \\ \boldsymbol{D} \end{pmatrix} \in \mathbb{C}^{g+1}, \quad \vec{b}^\epsilon = \begin{pmatrix} b_0^\epsilon \\ \boldsymbol{b}^\epsilon \end{pmatrix} \in H^1(\Sigma^\epsilon), \quad \vec{\omega}^\epsilon = \begin{pmatrix} \omega_0^\epsilon \\ \boldsymbol{\omega}^\epsilon \end{pmatrix}.$$

In the singular limit $\epsilon \to 0$ one has $\boldsymbol{b}^\epsilon \to \boldsymbol{b}$ and $\boldsymbol{\omega}^\epsilon \to \boldsymbol{\omega}$ which are the $\boldsymbol{b}$-cycles and the g holomorphic differentials on Σ respectively. The extra b-cycle b_0^ϵ becomes a degenerate cycle on the curve Σ, see figure 12. In appendix D we show that the extra holomorphic differential ω_0^ϵ on Σ^ϵ acquires a simple pole at the singular point and so becomes a normalised Abelian differential of the third kind. The Abelian differential $d\mathcal{Q}^\epsilon$ on Σ^ϵ is defined by the same singular parts (3.31) as $d\mathcal{Q}$ at $x = \pm 1$ but could potentially acquire an extra simple pole at the singular point. However, because $d\mathcal{Q}^\epsilon$ is normalised on Σ^ϵ, its residue there would vanish in the $\epsilon \to 0$ limit, so that in fact $d\mathcal{Q}^\epsilon \to d\mathcal{Q}$. One can also show that $C_i^\epsilon \to C_i$ as $\epsilon \to 0$.

The important object in (4.2) when considering the singular limit $\epsilon \to 0$ is the period matrix which can be broken down into blocks in a natural way

$$\tilde{\Pi}^\epsilon = \int_{\vec{b}^\epsilon} \vec{\omega}^\epsilon = \begin{pmatrix} \Pi_{00}^\epsilon & \boldsymbol{\Pi}_0^{\epsilon\,\mathsf{T}} \\ \boldsymbol{\Pi}_0^\epsilon & \Pi^\epsilon \end{pmatrix}.$$

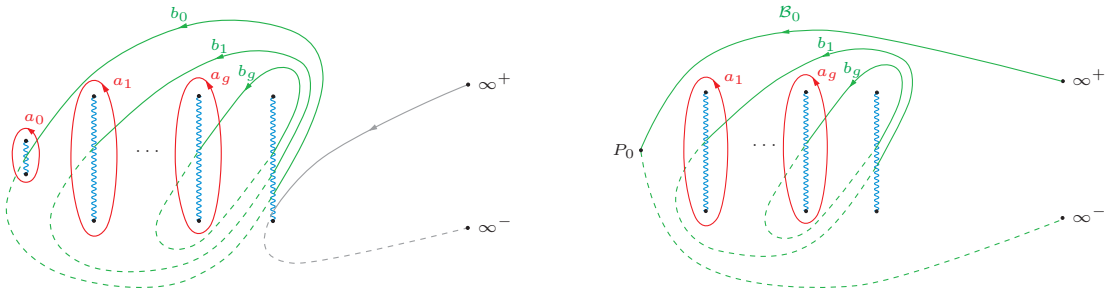
The singular limits of each block follow from the above considerations of $\vec{b}^\epsilon, \vec{\omega}^\epsilon$ in the limit (see appendix D for details). In particular, $\Pi^\epsilon \to \Pi$ as $\epsilon \to 0$ which is simply the period matrix of Σ. The vectors $\boldsymbol{\Pi}_0^\epsilon$ also stay finite in the limit. The top left component Π_{00}^ϵ on the other hand diverges in this limit, leading to a simplification of the Riemann θ-function $\theta(\cdot; \tilde{\Pi}^\epsilon)$ as $\epsilon \to 0$ which becomes expressible in terms of the Riemann θ-function $\theta(\cdot; \Pi)$ of Σ as in (D.3).

Now taking into account all the above limits and working to first order in ϵ, a direct computation shows that the difference δZ_i between expressions (4.2) and (4.1) contains three types of contribution

$$\delta Z_i = \Big(\{\text{periodic}\} + \{\text{periodic}\} \times e^{i\int_{b_0} d\mathcal{Q}} + \{\text{periodic}\} \times e^{-i\int_{b_0} d\mathcal{Q}}\Big) \times e^{\pi i \Pi_{00}^\epsilon}, \tag{4.3}$$

where "{periodic}" denotes functions periodic in all the angle variables φ_I of the underlying finite-gap solution (4.1), namely invariant under $\varphi_I \to \varphi_I + 2\pi$ for each $I = 1, \ldots, g+1$. The three contributions in (4.3) correspond to three different stability angles of the underlying solution (4.1) which can be read off directly

$$\nu_0^{(I)} = 0, \qquad \nu_\pm^{(I)} = \pm 2\pi \int_{b_0} dq^{(I)}, \quad I = 1, \ldots, g+1. \tag{4.4}$$



(a) (b)

Figure 13: The canonical cycles before (a) and after (b) shrinking of the 0^{th} cut. Note that it doesn't matter where the shrinking cut lies with respect to the other cuts, but for the sake of clarity of the figure we chose it to be the furthest to the left.

The zero stability angles $\nu_0^{(I)}$ are related to the φ_I-translation invariance of the equations of motion which is explicitly broken by the finite-gap solution (4.1).

Now stability angles are defined modulo 2π but for the underlying solution (4.1) to be periodic requires that

$$2\pi \int_{\infty^-}^{\infty^+} dq^{(I)} \in 2\pi\mathbb{Z}, \qquad I = 1, \ldots, g+1,$$

therefore we can redefine the stability angles $\nu_\pm^{(I)}$ as

$$\nu_\pm^{(I)} = \pm 2\pi \left(\int_{b_0} dq^{(I)} + \int_{\infty^+}^{\infty^-} dq^{(I)} \right) = \pm 2\pi \int_{\mathcal{B}_0} dq^{(I)}, \tag{4.5}$$

where the contour $\mathcal{B}_0$ runs from ∞^+ on the top sheet to ∞^- on the bottom sheet, by going through the 0^{th} cut, see figure 13. In the singular limit $\epsilon \to 0$ the 0^{th} cut shrinks to a point, say P_0 and so (4.5) yields

$$\begin{aligned} \nu_\pm^{(I)} &= \pm 2\pi \left(\int_{\infty^+}^{P_0} dq^{(I)} + \int_{P_0}^{\infty^-} dq^{(I)} \right) = \pm 2\pi \left(\int_{\infty^+}^{P_0} dq^{(I)} - \int_{P_0}^{\infty^-} \hat{\sigma}^* dq^{(I)} \right) \\ &= \pm 2\pi \left(\int_{\infty^+}^{P_0} dq^{(I)} - \int_{\hat{\sigma}P_0}^{\infty^+} dq^{(I)} \right) = \pm 2\pi \left(\int_{\infty^+}^{P_0} dq^{(I)} + \int_{\infty^+}^{\hat{\sigma}P_0} dq^{(I)} \right) \\ &\equiv \pm 2\pi \left(q^{(I)}(P_0) + q^{(I)}(\hat{\sigma}P_0) \right) = \pm 4\pi q^{(I)}(P_0), \end{aligned} \tag{4.6}$$

where $q^{(I)}(P) \equiv \int_{\infty^+}^{P} dq^{(I)}$ with the integral running along the top sheet (the precise choice of contour then doesn't matter since $dq^{(I)}$ is normalised) and the last equality follows from $P_0 = \hat{\sigma}P_0$ by virtue of P_0 being a singular point.

By repeating the calculation in (4.6) but for the $\mathcal{B}_0$-period of dp (the integrality of the b-periods of dp follows from the closed string requirement, namely that the finite-gap

JHEP06(2008)086

solution be periodic under $\sigma \to \sigma + 2\pi$),

$$\int_{\mathcal{B}_0} dp = 2\pi n_0, \quad n_0 \in \mathbb{Z},$$

the details of which can be found in [13], we arrive at an equation for the location of the singular point P_0, namely

$$p(P_0) = n_0\pi.$$

The above analysis shows that to this singular point P_0 there corresponds two stability angles for each of the $g+1$ cuts determined by the $\mathcal{B}_0$-period of corresponding quasi-action $dq^{(I)}$ or

$$\nu_{\pm}^{(I)} = \pm 4\pi q^{(I)}(P_0). \tag{4.7}$$

5. Semi-classical energy spectrum

As we recalled in section 3.2, every finite-gap solution to the equations of motion of a bosonic string on $\mathbb{R} \times S^3$ is constructed from a finite-genus algebraic curve Σ equipped with an additional set $\hat{\gamma}_0$ of $g+1$ points on it called a divisor (of degree $\deg \hat{\gamma}_0 = g+1$). This algebro-geometric data can be identified with a bundle $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ over the moduli space $\mathcal{L}$ of the algebraic curve Σ, of dimension $\dim_{\mathbb{C}} \mathcal{L} = g+1$,

$$S^{g+1}(\Sigma) \to \mathcal{M}_{\mathbb{C}}^{(2g+2)} \to \mathcal{L},$$

whose fibre over every point of the base, specifying a curve Σ, is the $(g+1)$-st symmetric product $S^{g+1}(\Sigma) = \Sigma^{g+1}/S_{g+1}$ of Σ (see [13] for moer details). The finite-gap construction of [13] defines an injective *geometric map,Krichever0* from this algebro-geometric data $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ into the space $\mathcal{S}_{\mathbb{C}}^{V}$ of complexified solutions $j \in \mathfrak{sl}(2,\mathbb{C})$ to the equations of motion of a string moving on $\mathbb{R} \times S^3$ which also satisfy the Virasoro and static gauge conditions,

$$\mathcal{G} : \mathcal{M}_{\mathbb{C}}^{(2g+2)} \hookrightarrow \mathcal{S}_{\mathbb{C}}^{V}. \tag{5.1}$$

Since a general point in phase-space is the restriction to the hypersurface $\tau = 0$ of the general solution we can identify the space $\mathcal{S}_{\mathbb{C}}$ of (complexified) solutions with (complexified) phase-space $\mathcal{P}_{\mathbb{C}}^{\infty}$. Furthermore, the subset $\mathcal{S}_{\mathbb{C}}^{V} \subset \mathcal{S}_{\mathbb{C}}$ of solutions satisfying Virasoro and static gauge conditions can be identified with the second class constraint surface $\mathcal{P}_{\mathbb{C}}^{V} \subset \mathcal{P}_{\mathbb{C}}^{\infty}$ defined by these conditions. We can describe the map (5.1) as an embedding $\mathcal{M}_{\mathbb{C}}^{(2g+2)} \hookrightarrow \mathcal{P}_{\mathbb{C}}^{V}$. If we further impose reality conditions by restricting the algebro-geometric data $\mathcal{M}_{\mathbb{C}}^{(2g+2)}$ to *real* algebro-geometric data (see [13] for a detailed discussion of reality conditions) then finite-gap integration describes an injective map [32]

$$\mathcal{G}_{\mathbb{R}} : \mathcal{M}_{\mathbb{R}}^{(2g+2)} \hookrightarrow \mathcal{P}_{\mathbb{R}}^{V}, \tag{5.2}$$

from the $(g+1)$-dimensional toric fibration $\mathbb{T}^{g+1} \to \mathcal{M}_{\mathbb{R}}^{(2g+2)} \to \mathcal{L}_{\mathbb{R}}$, with $\dim_{\mathbb{R}} \mathcal{L}_{\mathbb{R}} = g+1$, into the (real) phase-space $\mathcal{P}_{\mathbb{R}}^{V}$ of strings on $\mathbb{R} \times S^3$ satisfying the Virasoro and static gauge constraints. Introducing the inclusion $\iota_V : \mathcal{P}_{\mathbb{R}}^{V} \hookrightarrow \mathcal{P}_{\mathbb{R}}^{\infty}$ of the second class constraint surface

$\mathcal{P}^V_{\mathbb{R}} \subset \mathcal{P}^\infty_{\mathbb{R}}$, the Dirac bracket on $\mathcal{P}^V_{\mathbb{R}}$ is the pull-back of the symplectic structure ω on $\mathcal{P}^\infty_{\mathbb{R}}$. As was show in [14], the pull-back to $\mathcal{M}^{(2g+2)}_{\mathbb{R}}$ of this symplectic structure $\iota_V^*\omega$ on $\mathcal{P}^V_{\mathbb{R}}$ takes the simple form

$$\hat{\omega}_{2g+2} \equiv \mathcal{G}^*_{\mathbb{R}}\, \iota_V^*\omega = \sum_{I=1}^{g+1} \delta S_I \wedge \delta\varphi_I. \tag{5.3}$$

The different variables in (5.3) are defined as follows [14]:

- The *action variables* S_I are given by the filling fractions

$$S_I = \frac{1}{2\pi i}\int_{\mathcal{A}_I} \alpha, \quad I = 1, \ldots, g+1,$$

  where $\mathcal{A}_I$ is the cycle encircling the I^{th} cut $\mathcal{C}_I$ on the physical sheet of Σ represented as a hyperelliptic curve and $\alpha = \frac{\sqrt{\lambda}}{4\pi} z dp$ is a special 1-form on Σ, with $z \equiv x + \frac{1}{x}$ and $p(x)$ being the quasi-momentum.

- The *angle variables* φ_I are specified by the image of the divisor $\hat{\gamma}_0$ on the generalised Jacobian $J(\Sigma, \infty^\pm)$ of the curve Σ under the extended Abel map $\vec{\mathcal{A}} : S^{g+1}(\Sigma) \to J(\Sigma, \infty^\pm)$, or more precisely

$$\varphi_i = \mathcal{A}_i(\hat{\gamma}_0) - \mathcal{A}_{g+1}(\hat{\gamma}_0), \;\; i = 1, \ldots, g, \qquad \varphi_{g+1} = -\mathcal{A}_{g+1}(\hat{\gamma}_0).$$

The injective map (5.2) can thus be thought of as an embedding in phase-space of a $(g+1)$-parameter family of isotropic $(g+1)$-torii parametrised by $\{S_I\}_{I=1}^{g+1}$ since the pull-back (5.3) of the symplectic form ω to these torii is identically zero. This is the necessary set-up to apply the Bohr-Sommerfeld conditions (2.12) for the quantisation of a p-torus in an n-dimensional phase-space, where here the total phase-space is infinite dimensional so that $n = \infty$ and $p = g+1$. The condition (2.12) also involves the stability angles of perturbations around the p-torus which we computed in the section 4. So applying (2.12) to the finite-gap string we can write down the Bohr-Sommerfeld quantisation conditions for the action variables of the string as follows

$$\frac{S_I}{\hbar} = N_I + \frac{1}{2} + \sum_{\alpha=g+2}^{\infty} \left(n^I_\alpha + \frac{1}{2}\right) \frac{\nu^{(I)}_\alpha}{2\pi} + O(\hbar), \tag{5.4}$$

where we have used the fact that the Maslov index for the $\mathcal{A}_I$-cycle ($I = 1, \ldots, g+1$) in the generalised Jacobian $J(\Sigma, \infty^\pm)$ is simply $\mu_I = 2$. We emphasise that (5.4) is only valid in the harmonic oscillator approximation $N_I \gg n_\alpha$ where the perturbations are much smaller than the background filling fractions.

5.1 The main result

In the semiclassical regime, the Hamiltonian is defined by the same classical function of the actions $E_{\text{cl}}[S_1, \ldots, S_{g+1}]$ but evaluated on the action operators since by semiclassical integrability we have that $[\hat{S}_i, \hat{S}_j] = O(\hbar^3)$, so

$$\hat{\mathcal{H}}_{\text{string}} = E_{\text{cl}}[\hat{S}_1, \ldots, \hat{S}_{g+1}] + O(\hbar^2).$$

It follows that the energy spectrum is simply the classical energy $E_{\rm cl}$ evaluated on the eigenvalues of the action variables (5.4) namely

$$E = E_{\rm cl}\left[N_1\hbar + \frac{\hbar}{2} + \sum_{\alpha=g+2}^{\infty}\left(n_\alpha^1 + \frac{1}{2}\right)\frac{\nu_\alpha^{(1)}}{2\pi}\hbar, \ldots, \right.$$
$$\left. N_{g+1}\hbar + \frac{\hbar}{2} + \sum_{\alpha=g+2}^{\infty}\left(n_\alpha^{g+1} + \frac{1}{2}\right)\frac{\nu_\alpha^{(g+1)}}{2\pi}\hbar\right] + O(\hbar^2).$$

We now Taylor expand this using the fact that $N_I \gg n_\alpha$ to obtain

$$E = E_{\rm cl}\left[\left(N_1 + \frac{1}{2}\right)\hbar, \ldots, \left(N_{g+1} + \frac{1}{2}\right)\hbar\right] + \sum_{I=1}^{g+1}\sum_{\alpha=g+2}^{\infty}\left(n_\alpha^I + \frac{1}{2}\right)\frac{\partial E_{\rm cl}}{\partial S_I}\frac{\nu_\alpha^{(I)}}{2\pi}\hbar.$$

Using equations (3.40) and (4.5) to express $\partial E_{\rm cl}/\partial S_I$ and $\nu_\alpha^{(I)}$ respectively as $\mathcal{B}$-periods,

$$E = E_{\rm cl}\left[\left(N_1 + \frac{1}{2}\right)\hbar, \ldots, \left(N_{g+1} + \frac{1}{2}\right)\hbar\right] + \sum_{I=1}^{g+1}\sum_{\alpha=g+2}^{\infty}\left(n_\alpha^I + \frac{1}{2}\right)\int_{\mathcal{B}_I}\frac{dq}{2\pi}\int_{\mathcal{B}_\alpha} dq^{(I)}\hbar.$$

where $\mathcal{B}_\alpha$ is the contour running from ∞^+ to the singular point labelled α on the top sheet, and back on the bottom sheet to ∞^-. The sum over I can now be performed using equation (3.41) which yields

$$E = E_{\rm cl}\left[\left(N_1 + \frac{1}{2}\right)\hbar, \ldots, \left(N_{g+1} + \frac{1}{2}\right)\hbar\right] + \sum_{\alpha=g+2}^{\infty}\left(n_\alpha^I + \frac{1}{2}\right)\int_{\mathcal{B}_\alpha}\frac{dq}{2\pi}\hbar.$$

If we now formally think of the function $E_{\rm cl}$ as depending on the infinite set of filling fractions $\{S_I\}_{I=1}^{g+1}$, $\{S_\alpha\}_{\alpha=g+2}^{\infty}$ (all but finitely many of which are turned off for the classical finite-gap solutions) then we can interpret the $\mathcal{B}_\alpha$-period of $dq/2\pi$ as $\partial E_{\rm cl}/\partial S_\alpha$ using a formal analogue of (3.40) for an infinite gap solution. One can then resum the resulting Taylor expansion to obtain the following formal expression for the semiclassical energy spectrum

$$E = E_{\rm cl}\left[\left(N_1 + \frac{1}{2}\right)\hbar, \ldots, \left(N_{g+1} + \frac{1}{2}\right)\hbar, \left(n_{g+2} + \frac{1}{2}\right)\hbar, \ldots\right]. \tag{5.5}$$

We stress that this is only a formal derivation as rigorously one would have to regularise the divergent infinite sum over stability angles at the intermediate steps as well as subtract off the energy of the vacuum (i.e. the zero cut finite-gap solution). But formally at least the result of the above derivation is the following:

- The semiclassical energy spectrum is obtained by evaluating the classical energy function of an infinite-gap solution on filling fractions quantised to half-integer multiples of $\hbar$.
- The infinite number of singular points of the spectral curve $\det(\Omega(x) - y\mathbf{1}) = 0$ which accumulate at $x = \pm 1$ must be filled with half a unit of $\hbar$ in their ground state with an additional integer multiple of $\hbar$ for excitations.

JHEP06(2008)086

5.2 Comparison with alternative approach

In [16] an alternative method was proposed for extracting the semi-classical energy spacing around any given classical solution from the algebraic curve Σ itself, without making use of the divisor $\hat{\gamma}_0$ on Σ, and which the subsequent papers [17, 18] built upon. The heart of the method resides in the assumption that the filling fractions S_I become quantised in integer units at least in a semi-classical approximation. This assumption seems natural because the filling fractions constitute the action variables of the theory (a fact proved only in the $\mathbb{R} \times S^3$ subsector [14]) and we expect that after Bohr-Sommerfeld quantisation the action variables become half-integer multiples of $\hbar$. Yet we see from (5.4) that this is not the case. As we have argued at the start of this section, a finite-gap solution can be pictured as a degenerate isotropic $(g+1)$-torus within the full infinite dimensional phase-space. And although the Bohr-Sommerfeld conditions (2.9) for an integrable system do imply that the action variables become half-integer multiples of $\hbar$, we saw in section 2.2 and appendix B that these conditions receive corrections, when applied to a degenerate isotropic torus, from fluctuations transverse to the torus in the form of stability angles. It is only after the calculation in the previous subsection that one can effectively conclude that the semi-classical spectrum of a finite-gap solution amounts to evaluating its energy on a solution with filling fractions equal to half-integer multiples of $\hbar$.

Now the algebraic curve Σ is characterised by the quasi-momentum $p(x)$ used to define the filling fractions S_I as

$$S_I = \frac{1}{2\pi i}\frac{\sqrt{\lambda}}{4\pi}\int_{\mathcal{A}_I}\left(x+\frac{1}{x}\right)dp = -\frac{1}{2\pi i}\frac{\sqrt{\lambda}}{4\pi}\int_{\mathcal{A}_I}\left(1-\frac{1}{x^2}\right)p(x)dx, \quad I = 1,\ldots,g+1. \tag{5.6}$$

The integer quantisation of these filling fractions in the semiclassical limit can be interpreted in the language of the gauge theory side by attributing to a single Bethe root one unit of filling fraction. In the semiclassical quantisation of a solution each cut of its algebraic curve thus turns into a large clump of Bethe roots with the filling fraction counting the number of such roots [16]. The idea of [16] for obtaining the semiclassical energy spacings is then to compare the energies of two neighbouring classical solutions differing only by a single Bethe root. If the underlying solution is characterised by the quasi-momentum $p(x)$ and has K cuts $\mathcal{C}_j$ with mode numbers $n_j, j = 1,\ldots,K$,

$$p(x+i0)+p(x-i0) = 2\pi n_j, \quad x \in \mathcal{C}_j, j = 1,\ldots,K, \tag{5.7}$$

then its perturbation is characterised by a perturbed quasi-momentum $p(x)+\delta p(x)$ with still the same K cuts but also with an extra isolated Bethe root at x_{K+1} with mode number n_{K+1}

$$p(x+i0)+\delta p(x+i0)+p(x-i0)+\delta p(x-i0) = 2\pi n_j, \quad x \in \mathcal{C}_j, j = 1,\ldots,K, \tag{5.8a}$$

$$p(x_{K+1})+\delta p(x_{K+1})+p(x_{K+1})+\delta p(x_{K+1}) = 2\pi n_{K+1}. \tag{5.8b}$$

By using (5.7) we may simplify (5.8a) to

$$\delta p(x+i0)+\delta p(x-i0) = 0, \quad x \in \mathcal{C}_j, j = 1,\ldots,K. \tag{5.9a}$$

and since $\delta p(x)$ is small, by working to lowest order we can approximate (5.8b) as

$$p(x_{K+1}) = \pi n_{K+1}, \tag{5.9b}$$

Equations (5.9) are the starting point in [16] for obtaining the semiclassical energy spacings by reading them off from $\delta p(x)$.

Let us now show that the semiclassical energy spacings obtained by this method agrees with the semiclassical spectrum (5.5) obtained in the previous subsection. We know from (3.40) that the variation of the energy E of a classical solution as we vary the moduli S_I is

$$\delta E = \sum_{I=1}^{g+1} \left(\int_{\mathcal{B}_I} \frac{dq}{2\pi} \right) \delta S_I.$$

It follows that adding a single Bethe root (which would correspond to setting $\delta S_J = \hbar$ for some J) should increase the energy of the solution by

$$\delta E = \int_{\mathcal{B}_J} \frac{dq}{2\pi} \hbar. \tag{5.10}$$

This is exactly what one gets if we set $N_J \to N_J + 1$ in (5.5) and Taylor expand in the J^{th} entry using $N_J \gg 1$. We easily find that the energy evaluated on the solution with $S_J = N_J + 1$ is equal to the energy evaluated on the solution with $S_J = N_J$ plus the perturbation (5.10). Thus (5.5) predicts the same energy spacing (5.10) as we would expect if Bethe roots carried $\hbar$ units of filling fraction.

Note finally that the energy E_{cl} we have been using is not the space-time energy of the classical solution but rather the worldsheet energy or the Hamiltonian of the fields Z_i in the action. It can however be related to the space-time energy Δ by the following simple formula

$$E = \frac{\Delta^2}{2\sqrt{\lambda}}.$$

6. Summary and outlook

We have obtained the semiclassical energy spectrum of bosonic string theory on $\mathbb{R} \times S^3$ as expressed in equation (5.5) by semiclassically quantising the general finite-gap solution of this theory. The derivation of (5.5) can be summarised as follows. We have argued that the generic finite-gap solution can be thought of as an embedding of a $(g+1)$-torus Σ_f into the full infinite dimensional phase-space of the theory. Since these torii are finite-dimensional, they are all degenerate isotropic torii located on the boundary $\partial\mathcal{S}$ of the infinite region $\mathcal{S} \equiv \{S_I \geq 0, \forall I\}$. But a procedure due to Voros [22, 23] provides a way of semiclassically quantising such degenerate torii: the method consists of studying neighbouring orbits in the small oscillator approximation, which would live on a neighbouring non-degenerate torus in the interior of $\mathcal{S}$, and then quantise this torus in the usual way using Bohr-Sommerfeld-Maslov quantisation conditions. The computation in section 5.1 consisted in formally rewriting the quantised energy of such a linearised torus in terms of the energy of

the infinite-gap solution it is approximating in the interior of $\mathcal{S}$ (but still near $\partial\mathcal{S}$). The result is that the semiclassical energy spectrum can be obtained by evaluating the classical energy function on points of the following infinite lattice in $\mathcal{S}$,

$$S_I \in \hbar\left(\frac{1}{2}+\mathbb{N}\right), \forall I. \tag{6.1}$$

Yet because we computed the semiclassical spectrum around finite-gap solutions which only describe $\partial\mathcal{S}$, it follows that (5.5) only describes this lattice structure near the boundary of $\mathcal{S}$ where the n's are much smaller than the finitely many N's in (5.5). However, since the number of N's is finite but arbitrary, by formally considering the infinite genus limit of finite-gap solutions, the complete spectrum is described by $E = E_{\rm cl}\left[\left(N_1+\frac{1}{2}\right)\hbar, \ldots\right]$ so that (6.1) should give the correct lattice structure in the whole bulk of $\mathcal{S}$.

Such a procedure for semiclassically quantising finite-gap solutions should be sufficiently general to apply with little modification to more general settings and in particular to the case of superstrings on $AdS_5 \times S^5$. As stated in the introduction, it would therefore be very interesting to obtain the divisor for the full algebraic curve of $AdS_5 \times S^5$ by constructing the finite-gap solution in full generality on $AdS_5 \times S^5$.

Finally, in view of ultimately obtaining an exact quantisation of string theory on $AdS_5 \times S^5$ we have argued that operator ordering issues will be of crucial importance since they already appear in the semiclassical analysis. In this paper we assumed for simplicity that the cohomology class of the subprincipal form vanished since with this assumption we were able to reproduce the semiclassical spectrum of [16 – 18] at least for the fluctuations in the $\mathbb{R} \times S^3$ subspace. This rules out many operator orderings for the exact quantisation, namely all those for which the action variables have a subprincipal Weyl symbol.

Acknowledgments

I am very grateful to Nick Dorey for many useful insights and discussions on semiclassical quantisation as well as for following the details of this work very closely. I would also like to thank Harry Braden for interesting discussions on finite-gap integration.

A. Symbolic calculus of pseudo-differential operators

The passage from a classical system on phase-space T^*X to its quantum counterpart involves promoting the algebra of classical observable $C(T^*X)$ to a noncommutative algebra $\mathcal{A}$ of operators. Classically, the Poisson algebra of observables is uniquely specified by the choice of a symplectic structure $\omega = \sum_i dx_i \wedge d\xi_i$ and the Poisson bracket of two observables $f, g \in C(T^*X)$ is then defined by $\{f, g\} = \omega(X_f, X_g)$, where X_H denotes the Hamiltonian vector field associated to any function $H \in C(T^*X)$ satisfying $i_{X_H}\omega = dH$. To pass to quantum mechanics, the prescription of *canonical quantisation* is to promote the special functions $x_i, \xi_i \in C(T^*X)$ to operators $\hat{x}_i, \hat{\xi}_i$ and the symplectic structure $\omega = \sum_i dx_i \wedge d\xi_i$ to the Weyl algebra $[\hat{x}_i, \hat{\xi}_j] = i\hbar\delta_{ij}$ which admits the unique representation $\hat{x}_i = x_i, \hat{\xi}_i = -i\hbar\partial/\partial x_i \equiv -i\hbar\partial_i$ in terms of differential operators on $L^2(X)$. The

problem that remains after canonical quantisation is to associate with any other given observable $f \in C(T^*X)$ (which is a function of x_i, ξ_i) a (pseudo-)differential operator $\hat{f}$ on $L^2(X)$, and it is immediately obvious that this is by no means unique. Many different operators correspond to the same classical function: for instance, given any $t \in \mathbb{R}$, the differential operator $tx_1\partial_1 + (1-t)\partial_1 \cdot x_1$ is a possible candidate for the quantisation of the function $x_1\xi_1$. In other words, it is not possible to specify the operator ordering in an operator $\hat{f}$ starting from just single function $f \in C(T^*X)$. However, with an infinite set of functions $f_k \in C(T^*X)$ it turns out to be possible to associate a unique operator $\hat{f}$ by canonical quantisation. Such a set defines a function of $\hbar$ through the asymptotic expansion

$$f(x, \xi; \hbar) \underset{\hbar \to 0}{\sim} \sum_{k \geq 0} f_k(x, \xi)\hbar^k. \tag{A.1}$$

We refer to such a $\hbar$-dependent function $f(\hbar) \in C(T^*X)$ as a *classical (Weyl) symbol*, which is technically required to satisfy certain estimates, such as all its partial derivatives being uniformly bounded by some order function.

Without going into details of the construction, we now state the map from symbols to *pseudo-differential operators*[4] (ΨDO for short). Given a symbol $f(\hbar)$, we define the corresponding ΨDO by specifying its action on $u \in L^2(X)$ using the *Weyl quantisation* formula [27]

$$\left(\mathrm{Op}^W_\hbar(f(\hbar))u\right)(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(x-y)\cdot\xi} f\left(\frac{x+y}{2}, \xi; \hbar\right) u(y) dy d\xi.$$

It is important to note here that the choice of Weyl quantisation in the definition of the ΨDO from its symbol does not limit us to having only Weyl ordered ΨDOs. Indeed, the operator $\mathrm{Op}^W_\hbar(f(\hbar))$ is Weyl ordered only when the corresponding Weyl symbol is $\hbar$-independent. So it is precisely the subleading terms in the asymptotic expansion (A.1) of the symbol $f(x, \xi; \hbar)$ which account for the different possible choices of orderings in the definition of the ΨDO. For example, the Weyl ordered operator of the classical observable $x_1\xi_1$ is given simply by the Weyl symbol $x_1\xi_1$, namely

$$\mathrm{Op}^W_\hbar(x_1\xi_1) = \frac{-i\hbar}{2}\left(x_1\partial_1 + \partial_1 \cdot x_1\right),$$

whereas the left ordered operator $-i\hbar x_1\partial_1$ which corresponds to the same classical observable $x_1\xi_1$ as $\mathrm{Op}^W_\hbar(x_1\xi_1)$ is given by a Weyl symbol with a subleading term in $\hbar$ since

$$\mathrm{Op}^W_\hbar\left(x_1\xi_1 + \frac{i\hbar}{2}\right) = -i\hbar x_1\partial_1.$$

Naturally the right ordered operator $-i\hbar\partial_1 \cdot x_1$ has Weyl symbol $x_1\xi_1 - \frac{i\hbar}{2}$. A general ΨDO A always has a unique Weyl symbol, which is a $\hbar$-dependent function $f(x, \xi; \hbar)$ denoted $\sigma^W(A)$. The leading non-zero term in the asymptotic expansion (A.1) of this Weyl symbol

[4] When the symbol $f(x, \xi; \hbar)$ is a polynomial in x, ξ the associated operator is an ordinary partial differential operator. To include the more general case when $f(x, \xi; \hbar)$ might not be a polynomial we talk about pseudo-differential operators.

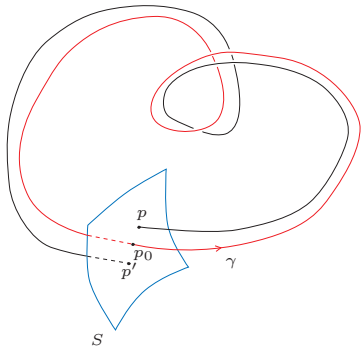


Figure 14: Poincaré map: global perturbations of a periodic orbit γ can be studied locally in terms of a map $\psi : S \to S$ defined by the flow of the Hamiltonian vector field X_H.

is called the *principal symbol*, denoted $\sigma_0^W(A)$, and the subleading term is called the *subprincipal symbol*, denoted $\sigma_{\rm sub}^W(A)$. For instance, if $f_0(x,\xi) \neq 0$ then $\sigma_0^W(A) = f_0(x,\xi)$ and $\sigma_{\rm sub}^W(A) = f_1(x,\xi)\hbar$.

An important object for the study of quantum integrability is the commutator $[A,B]$ of two operators A and B. In the present context of ΨDOs one can show that if A, B are ΨDOs then their commutator $[A,B]$ is also a ΨDO with principal symbol

$$\sigma_0^W([A,B]) = -i\hbar\left\{\sigma_0^W(A), \sigma_0^W(B)\right\},$$

(so that $-i\hbar\sigma_0^W$ is a Lie algebra homomorphism) and subprincipal symbol

$$\sigma_{\rm sub}^W([A,B]) = -i\hbar\left\{\sigma_0^W(A), \sigma_{\rm sub}^W(B)\right\} - i\hbar\left\{\sigma_{\rm sub}^W(A), \sigma_0^W(B)\right\}.$$

B. Bohr-Sommerfeld for isolated periodic orbit

Let γ be a given periodic orbit of energy E, i.e. $\gamma \subset \Sigma_E$. We henceforth assume that E is a regular value of H so that Σ_E is a smooth codimension one submanifold of T^*X. Given a point $p_0 \in \gamma$, we call a section of γ at p_0 a smooth codimension one surface $S \subset \Sigma_E$ transverse to γ and intersecting it at p_0. We then define the local map $\psi : S \to S$ near p_0 by letting $p' = \psi(p)$ be the unique point obtained by following $p \in S$ around the Hamiltonian flow X_H for a time close to the period T_γ of γ (see figure 14). Note that fixed points $p = \psi(p)$ (respectively periodic points $p = \psi^k(p), k \geq 2$) of ψ correspond to periodic orbits of the Hamiltonian flow X_H of period close to T_γ (respectively close to kT_γ). In particular, since $p_0 = \psi(p_0)$ we define the Poincaré map as the differential of ψ at p_0 [25]

$$P = d\psi_{p_0} : T_{p_0}S \to T_{p_0}S.$$

We say that the periodic orbit γ is non-degenerate if and only if 1 is not an eigenvalue of the Poincaré map. This is a way of saying that γ is isolated on Σ_E in the sense that there are no periodic orbits on Σ_E arbitrarily close to it. However, although γ is isolated on Σ_E, it belongs to a continuous 1-parameter family γ_E of periodic orbits intersecting Σ_E at γ(see figure 2). This is the content of the "cylinder theorem" (see for instance [25, 26]).

JHEP06(2008)086

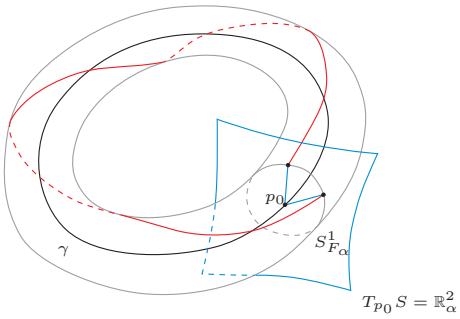


Figure 15: The infinitesimal torus around a stable isolated periodic orbit γ ($p = 1$) illustrated in the case $n = 2$ where there is only one stability angle ν_α and $T_{p_0}S = \mathbb{R}^2_\alpha$.

If γ is stable, then the eigenvalues of the Poincaré map defined at a point $p_0 \in \gamma$ come in complex conjugate pairs of the form $(e^{i\nu_\alpha}, e^{-i\nu_\alpha}), \nu_\alpha \in \mathbb{R}$ and hence the Poincaré map is merely a product of rotations by angles ν_α in $n-1$ disjoint planes $\mathbb{R}^2_\alpha \subset T_{p_0}S$. In other words, every point $p_0 \in \gamma$ of the stable isolated periodic orbit γ is surrounded by an infinitesimal torus $S^1_{F_2} \times \ldots \times S^1_{F_n}$, where $S^1_{F_\alpha} = \{x_\alpha \in \mathbb{R}^2_\alpha \mid ||x_\alpha||^2 = F_\alpha\} \subset \mathbb{R}^2_\alpha$, which is preserved by the Poincaré map to first approximation in $F_\alpha \ll 1$. By the cylinder theorem the periodic orbit γ belongs to a continuous family γ_E parametrised by the energy E, and so one could now apply the Bohr-Sommerfeld-Maslov quantisation conditions to the family of torii $\Lambda \equiv \gamma_E \times S^1_{F_2} \times \ldots \times S^1_{F_n}$ just constructed (see figure 15)

$$\int_{S^1_{F_\alpha}} \alpha = 2\pi \left(n_\alpha + \frac{1}{2}\right) \hbar + O(\hbar^2), \quad \alpha = 2, \ldots, n$$
$$\int_{\tilde{\gamma}} \alpha = 2\pi \left(N + \frac{\mu_\gamma}{4}\right) \hbar + O(\hbar^2),$$

where $\tilde{\gamma}$ is the closed path on Λ consisting of a classical path going from $T_{p_0}S$ once around Λ back to $T_{p_0}S$ and the set of arcs of angles $-\nu_\alpha$ on $T_{p_0}S$ to close off this classical path (see red curve in figure 15).

Consider the 2-dimensional surface Γ bounded by the periodic orbit γ and the closed curve $\tilde{\gamma}$, constructed in the obvious way: at any point $t \neq 0$ along the curve $\gamma(t)$, Γ looks locally like $\{\gamma(t) + \tau y(t) | 0 < t < T, 0 \leq \tau \leq 1\}$ where $y(t)$ is the transversal vector to γ joining the points $\gamma(t)$ and $\tilde{\gamma}(t)$. At $t = 0$ we complete the surface by adding the sections of the disc of angle $-\nu_\alpha$ on $T_{p_0}S$. Then by Stokes's theorem we have

$$\left(\int_{\tilde{\gamma}} - \int_{\gamma}\right) \alpha = \int_{\partial\Gamma} \alpha = \int_{\Gamma} \omega.$$

On the part of Γ corresponding to $t \neq 0$ we have $\omega|_\Gamma = 0$ since the tangent space to Γ is spanned by X_H and the transversal vector y ($i_y i_{X_H}\omega = i_y dH = y(H) = 0$ since y lies in the energy surface Σ_E). And since $\Gamma_{t=0}$ looks like sections of angle $-\nu_\alpha$ of the disc of radius $\sqrt{F_\alpha}$ it follows that

$$\left(\int_{\tilde{\gamma}} - \int_{\gamma}\right) \alpha = \int_{\Gamma_{t=0}} \omega = -\sum_{\alpha=2}^{n} \nu_\alpha F_\alpha.$$

JHEP06(2008)086

On the other hand we have that

$$\int_{S^1_{F_\alpha}} \alpha = \int_{D^1_{F_\alpha}} \omega = 2\pi F_\alpha,$$

where $D^1_{F_\alpha}$ is the disc in $\mathbb{R}^2_\alpha$ bounded by the circle $S^1_{F_\alpha}$. The last equality follows by a direct computation, in analogy with the harmonic oscillator. Finally, by combining all the above we obtain [22, 23]

$$\int_\gamma \alpha = \left[2\pi\left(N + \frac{\mu_\gamma}{4}\right) + \sum_{\alpha=2}^{n}\left(n_\alpha + \frac{1}{2}\right)\nu_\alpha\right]\hbar + O(\hbar^2), \tag{B.1}$$

which is just the Bohr-Sommerfeld condition for an isolated periodic orbit.

C. Dirac brackets

Just as in [14], in this paper we work in conformal static gauge in order to isolate the physical degrees of freedom of the string. This is done by imposing the Virasoro constraints and static gauge fixing condition. However, these constraints together form a set of second class constraints and so to consistently impose these constraints from the outset one must work with Dirac brackets instead of Poisson brackets. In this section we show that the for the type of brackets $\{\operatorname{tr}\Omega(x), \cdot\}$ considered in section 3 this distinction does not matter since

$$\{\operatorname{tr}\Omega(x), f\}_{\text{D.B.}} = \{\operatorname{tr}\Omega(x), f\}_{\text{P.B.}}$$

for an arbitrary function f of the principal chiral model fields $j = -g^{-1}dg$ and so by abuse of notation we drop the suffices on both brackets and write $\{\cdot,\cdot\}$ throughout section 3.

We start with the Poisson bracket (3.8). To compute Poisson brackets on the circle we shall work on the universal cover $\mathbb{R}$. So let $\sigma_1 = \sigma + 2\pi$, $\sigma_2 = \sigma$ and $\sigma_3 = \sigma'$ in (3.8) to obtain the Poisson bracket $\{\Omega(\sigma, x)\overset{\otimes}{,}J_1(\sigma', x')\}$. This easily leads to the Poisson brackets $\{\Omega(\sigma, x)\overset{\otimes}{,}j_\pm(\sigma')\}$ after noting from the definition of $J_1(x)$ that $J_1(0) = \frac{1}{2}(j_+ - j_-)$ and $\lim_{x\to\infty}(-x)J_1(x) = \frac{1}{2}(j_+ + j_-)$, in particular

$$\begin{aligned}\{\Omega(\sigma, x)\overset{\otimes}{,}j_\pm(\sigma')\}_{\text{P.B.}} = (T(\sigma+2\pi, \sigma', x)\otimes\mathbf{1}) \times \Bigg((\delta(\sigma'-\sigma-2\pi)-\delta(\sigma'-\sigma))\frac{4\pi}{\sqrt{\lambda}}\frac{1\pm x}{1-x^2}\eta \\ +\chi(\sigma'; \sigma+2\pi, \sigma)\left[-\frac{2\pi}{\sqrt{\lambda}}\frac{2x}{1-x^2}\eta, (x\pm 1)J_1(\sigma', x)\otimes\mathbf{1} \pm \mathbf{1}\otimes\frac{1}{2}(j_+(\sigma')-j_-(\sigma'))\right]\Bigg) \\ \times(T(\sigma', \sigma, x)\otimes\mathbf{1}),\end{aligned}$$

where we have used the definitions of the r, s-matrices [14] which involve the tensor product $\eta = \frac{1}{2}\sigma_a\otimes\sigma_a$. Using the identity $\operatorname{tr}_2(\eta\mathbf{1}\otimes A) = A$ for any matrix $A \in \mathfrak{su}(2)$ one can show that after multiplying the above equation by $\mathbf{1}\otimes j_\pm(\sigma')$ and taking the trace tr_2 over the second tensor factor the commutator disappears and we are left with

$$\left\{\Omega(\sigma, x), \frac{1}{2}\operatorname{tr} j^2_\pm(\sigma')\right\}_{\text{P.B.}} = \frac{4\pi}{\sqrt{\lambda}}(\delta(\sigma'-\sigma-2\pi)-\delta(\sigma'-\sigma))T(\sigma+2\pi, \sigma', x)J_\pm(\sigma', x)T(\sigma', \sigma, x),$$

where $J_\pm(\sigma', x) = j_\pm(\sigma')/(1 \mp x)$. Next we multiply both sides by $e^{\pm in\sigma'}$ and integrate over σ' from 0 to 2π. However, since we are on the universal cover $\mathbb{R}$ of S^1 we get two non-zero contributions, namely from the integrations over the two lifts $[0, 2\pi]$ and $[2\pi, 4\pi]$ (assuming $\sigma \in (0, 2\pi)$). Definition the Virasoro generators

$$L_n = \frac{\sqrt{\lambda}}{8\pi} \int_0^{2\pi} d\sigma' e^{in\sigma'} \frac{1}{2} j_+^2(\sigma'), \quad \tilde{L}_n = \frac{\sqrt{\lambda}}{8\pi} \int_0^{2\pi} d\sigma' e^{-in\sigma'} \frac{1}{2} j_-^2(\sigma'),$$

we can write the result as follows

$$\{\Omega(\sigma, x), L_n\}_{\text{P.B.}} = \frac{1}{2} e^{in\sigma} [J_+(\sigma, x), \Omega(\sigma, x)], \quad \{\Omega(\sigma, x), \tilde{L}_n\}_{\text{P.B.}} = \frac{1}{2} e^{-in\sigma} [J_-(\sigma, x), \Omega(\sigma, x)].$$

Note that in the above calculation it is because of the presence of the s-matrix, which arises from non-ultralocality of the Poisson brackets of the model, that we end up with the correct transformation property for $\Omega(x)$ under conformal transformations. Finally, since the right hand sides are commutators, taking the trace shows that $\operatorname{tr} \Omega(x)$ is invariant under conformal transformations generated by $L_n, \tilde{L}_n$, namely

$$\{\operatorname{tr} \Omega(x), L_n\}_{\text{P.B.}} = \{\operatorname{tr} \Omega(x), \tilde{L}_n\}_{\text{P.B.}} = 0.$$

The assertion that the Dirac and Poisson brackets involving the quantity $\operatorname{tr} \Omega(x)$ are equal now follows from the definition of the Dirac bracket which in the present case reads,

$$\begin{aligned} \{\operatorname{tr} \Omega(x), f\}_{\text{D.B.}} = {} & \{\operatorname{tr} \Omega(x), f\}_{\text{P.B.}} - \{\operatorname{tr} \Omega(x), L_n\}_{\text{P.B.}} \{L_n, L_m\}^{-1}_{\text{P.B.}} \{L_m, f\}_{\text{P.B.}} \\ & - \{\operatorname{tr} \Omega(x), \tilde{L}_n\}_{\text{P.B.}} \{\tilde{L}_n, \tilde{L}_m\}^{-1}_{\text{P.B.}} \{\tilde{L}_m, f\}_{\text{P.B.}}, \end{aligned}$$

for any function f of the principal chiral model fields $j = -g^{-1}dg$.

D. Pinching an a-period

In this appendix we determine the behaviour of the Riemann θ-function when the underlying algebraic curve Σ becomes singular [37, 38]. To determine the effect of degenerating an a-cycle on the algebraic curve Σ, let us consider a family Σ^ϵ of Riemann surfaces ($\epsilon > 0$) of genus $g+1$ with homology basis $\{a_i^\epsilon, b_i^\epsilon\}_{i=0}^g$ of $H_1(\Sigma^\epsilon, \mathbb{R})$. Let $\{\omega_i^\epsilon\}_{i=0}^g$ be a dual basis of holomorphic 1-forms canonically normalised as

$$\int_{a_j^\epsilon} \omega_k^\epsilon = \delta_{jk}. \tag{D.1}$$

We model the pinching of an a-cycle of the algebraic curve Σ by choosing a family $\{\Sigma^\epsilon\}_{\epsilon>0}$ for which a particular marked cycle $\tilde{a}_0^\epsilon$ on Σ^ϵ homotopic to a_0^ϵ shrinks to a point P_0 in the singular limit $\epsilon \to 0$. The resulting surface Σ^0 is singular at P_0, and we denote by Σ' its desingularisation (see figure 16).

In the limit $\epsilon \to 0$ where the cycle $\tilde{a}_0^\epsilon$ shrinks to a single point $P_0 \in \Sigma^0$, the cycles a_0 and a'_0 are homotopic to the punctures P_0^+ and P_0^- on Σ' corresponding to the desingularisation

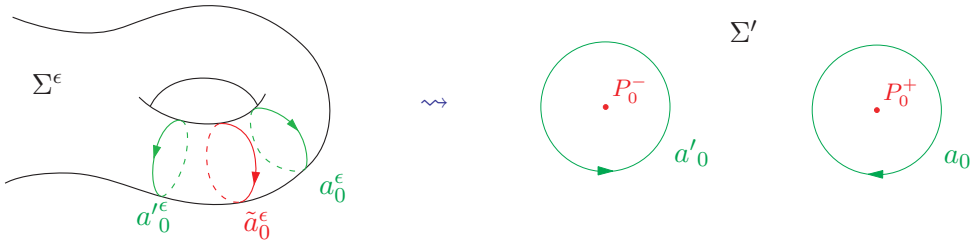


Figure 16: The curve Σ' is the result of pinching the cycle $\tilde{a}_0$ followed by a desingularisation. It has two punctures $P_0^\pm$ with cycles a_0 and a'_0 homotopic to these punctures respectively.

of P_0 on Σ^0. It follows from (D.1) that in the limit $\epsilon \to 0$ the 1-form ω_0^ϵ acquires simple poles at the pair of points $P_0^\pm$ with residues

$$\mathrm{res}_{P_0^+}\omega_0 = \frac{1}{2\pi i}\int_{a_0}\omega_0 = \frac{1}{2\pi i}, \qquad \mathrm{res}_{P_0^-}\omega_0 = \frac{1}{2\pi i}\int_{a'_0}\omega_0 = -\frac{1}{2\pi i}.$$

Since ω_0 has no further poles it is a normalised ($\int_{a_i}\omega_0 = 0$, $i = 1, \ldots, g$) Abelian differential of the third kind on Σ'. Moreover, $\{\omega_i\}_{i=1}^g$ is a basis of holomorphic 1-forms on Σ' dual to the homology basis $\{a_i, b_i\}_{i=1}^g$ for Σ'. Since the curve b_0 starts and ends at $P_0^\pm$, the component $\Pi_{00}^\epsilon = \int_{b_0^\epsilon}\omega_0^\epsilon$ of the period matrix will blow up as $\epsilon \to 0$. All other components of the period matrix $\Pi_{ij}^\epsilon = \int_{b_i^\epsilon}\omega_j^\epsilon$ and $\Pi_{0j}^\epsilon = \int_{b_0^\epsilon}\omega_j^\epsilon$ stay finite in the limit $\epsilon \to 0$. The behaviour of the Riemann θ-function associated with Σ^ϵ

$$\theta(\vec{z};\tilde{\Pi}^\epsilon) = \sum_{\vec{m}\in\mathbb{Z}^{g+1}} \exp\left\{i\langle\vec{m},\vec{z}\rangle + \pi i\langle\tilde{\Pi}^\epsilon\vec{m},\vec{m}\rangle\right\} \tag{D.2}$$

can now be analysed in the limit $\epsilon \to 0$. Using the fact that the imaginary part $\mathrm{Im}\,\Pi$ of the period matrix Π is positive definite we have $\mathrm{Im}\,\Pi_{00} = \mathrm{Im}\,\langle\Pi e^{(0)}, e^{(0)}\rangle > 0$. It follows that the quantity $e^{\pi i\Pi_{00}^\epsilon}$ is vanishingly small in the limit $\epsilon \to 0$ and one finds that (D.2) can be expanded as follows

$$\theta(\vec{z};\tilde{\Pi}^\epsilon) = \theta(\boldsymbol{z};\Pi^\epsilon) + \left[\theta(\boldsymbol{z}+\boldsymbol{\Pi}_0^\epsilon;\Pi^\epsilon)e^{iz_0} + \theta(\boldsymbol{z}-\boldsymbol{\Pi}_0^\epsilon;\Pi^\epsilon)e^{-iz_0}\right]e^{\pi i\Pi_{00}^\epsilon} + O\left(e^{2\pi i\Pi_{00}^\epsilon}\right) \tag{D.3}$$

where

$$\vec{z} = \begin{pmatrix} z_0 \\ \boldsymbol{z}\end{pmatrix} \in \mathbb{C}^{g+1}, \quad \tilde{\Pi}^\epsilon = \begin{pmatrix} \Pi_{00}^\epsilon & \boldsymbol{\Pi}_0^{\epsilon\,\mathsf{T}} \\ \boldsymbol{\Pi}_0^\epsilon & \Pi^\epsilon\end{pmatrix}.$$

References

[1] R.F. Dashen, B. Hasslacher and A. Neveu, *Nonperturbative methods and extended hadron models in field theory. 1. Semiclassical functional methods*, *Phys. Rev.* **D 10** (1974) 4114.

[2] R.F. Dashen, B. Hasslacher and A. Neveu, *Nonperturbative methods and extended hadron models in field theory. 2. Two-dimensional models and extended hadrons*, *Phys. Rev.* **D 10** (1974) 4130.

[3] R.F. Dashen, B. Hasslacher and A. Neveu, *Nonperturbative methods and extended hadron models in field theory. 3. Four-dimensional nonabelian models*, *Phys. Rev.* **D 10** (1974) 4138.

[4] V.E. Korepin, P.P. Kulish and L.D. Faddeev, *Soliton quantization*, *JETP Lett.* **21** (1975) 138 [*Pisma Zh. Eksp. Teor. Fiz.* **21** (1975) 302].

[5] V.E. Korepin and L.D. Faddeev, *Quantization of solitons*, *Theor. Math. Phys.* **25** (1975) 1039 [*Teor. Mat. Fiz.* **25** (1975) 147].

[6] M.V. Berry and M. Tabor, *Closed orbits and the regular bound spectrum*, *Proc. Roy. Soc. Lond.* **A 349** (1976) 101.

[7] M.V. Berry and M. Tabor, *Calculating the bound spectrum by path summation in action angle variables*, *J. Phys.* **A 10** (1977) 371.

[8] S.R. Coleman, *Classical lumps and their quantum descendents*, PRINT-77-0088.

[9] R. Rajaraman, *Solitons and instantons. An introduction to solitons and instantons in quantum field theory*, North-holland, Amsterdam The Netherlands (1982).

[10] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[11] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116].

[12] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207];
N. Beisert, V.A. Kazakov and K. Sakai, *Algebraic curve for the* SO(6) *sector of AdS/CFT*, *Commun. Math. Phys.* **263** (2006) 611 [hep-th/0410253];
N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$*, *Commun. Math. Phys.* **263** (2006) 659 [hep-th/0502226].

[13] N. Dorey and B. Vicedo, *On the dynamics of finite-gap solutions in classical string theory*, *JHEP* **07** (2006) 014 [hep-th/0601194].

[14] N. Dorey and B. Vicedo, *A symplectic structure for string theory on integrable backgrounds*, *JHEP* **03** (2007) 045 [hep-th/0606287].

[15] B. Vicedo, *Giant magnons and singular curves*, *JHEP* **12** (2007) 078 [hep-th/0703180].

[16] N. Gromov and P. Vieira, *The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve*, *Nucl. Phys.* **B 789** (2008) 175 [hep-th/0703191].

[17] N. Gromov and P. Vieira, *Constructing the AdS/CFT dressing factor*, *Nucl. Phys.* **B 790** (2008) 72 [hep-th/0703266].

[18] N. Gromov and P. Vieira, *Complete* 1*-loop test of AdS/CFT*, *JHEP* **04** (2008) 046 [arXiv:0709.3487].

[19] J.M. Maillet, *New integrable canonical structures in two-dimensional models*, *Nucl. Phys.* **B 269** (1986) 54; *Kac-Moody algebra and extended Yang-Baxter relations in the O(N) non-linear σ-model*, *Phys. Lett.* **B 162** (1985) 137; *Hamiltonian structures for integrable classical theories from graded Kac-Moody algebras*, *Phys. Lett.* **B 167** (1986) 401;
M. Bordemann, M. Forger, J. Laartz and U. Schäper, *The Lie-Poisson structure of integrable classical non-linear sigma model*, *Commun. Math. Phys.* **152** (1993) 167.

[20] S. Bates and A. Weinstein, *Lectures on the geometry of quantization*, AMS, Providence U.S.A. (1997).

JHEP06(2008)086

[21] A. Grigis and J. Sjostrand, *Microlocal analysis for differential operators: an introduction*, London Mathematical Society Lecture Note Series **196**, Cambridge University Press, Cambridge U.K. (1994).

[22] A. Voros, *The WKB-Maslov method for nonseparable systems*, Géométrie symplectique et physique mathématique, Aix-en-Provence France (1974).

[23] A. Voros, *Semiclassical approximations*, *Ann. Inst. Henri Poincaré* **24** (1976) 31 [SACLAY-D.PH-T-74-57].

[24] A. Voros, *The WKB method in the Bargmann representation*, *Phys. Rev.* **A 40** (1989) 6814 [NSF-ITP-89-104].

[25] J. Moser and E. Zehnder, *Notes on dynamical systems*, Courant Lecture Notes, American Mathematical Society U.S.A. (2005).

[26] R. Abraham and J.E. Marsden, *Foundations of mechanics*, Addison Wesley Publishing Company, 2nd edition (1980).

[27] A. Martinez, *An introduction to semiclassical and microlocal analysis*, Universitext, Springer, Germany (2002).

[28] S. Vũ Ngọc, *Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type*, *Commun. Pure Appl. Math.* **53** (2000) 143.

[29] S. Vũ Ngọc, *Quantum monodromy and Bohr-Sommerfeld rules*, *Lett. Math. Phys.* **55** (2001) 205.

[30] E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its and V.B. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, Springer-Verlag Telos (1994).

[31] O. Babelon, D. Bernard and M. Talon, *Introduction to classical integrable systems*, Cambridge University Press, Cambridge U.K. (2003).

[32] I.M. Krichever and D.H. Phong, *Symplectic forms in the theory of solitons*, `hep-th/9708170`.

[33] I.M. Krichever and D.H. Phong, *On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories*, *J. Diff. Geom.* **45** (1997) 349 [`hep-th/9604199`].

[34] I.M. Krichever, *Methods of algebraic geometry in the theory of non-linear equations*, *Russ. Math. Surveys* **32** (1977) 185.

[35] I.M. Krichever, *Integration of non-linear equations by methods of algebraic geometry*, *Funct. Anal. Appl.* **11** (1977) 12.

[36] I. Krichever, *Vector bundles and Lax equations on algebraic curves*, *Commun. Math. Phys.* **229** (2002) 229 [`hep-th/0108110`].

[37] J.D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, Springer-Verlag, Germany (1973).

[38] H.P. McKean, *Integrable systems and algebraic curves*, in *Global analysis*, Lecture Notes in Mathematics, Springer-Verlag, Germany (1979).

JHEP06(2008)086